Andrew Glassner's Notebook

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Celtic Knots, Part 3

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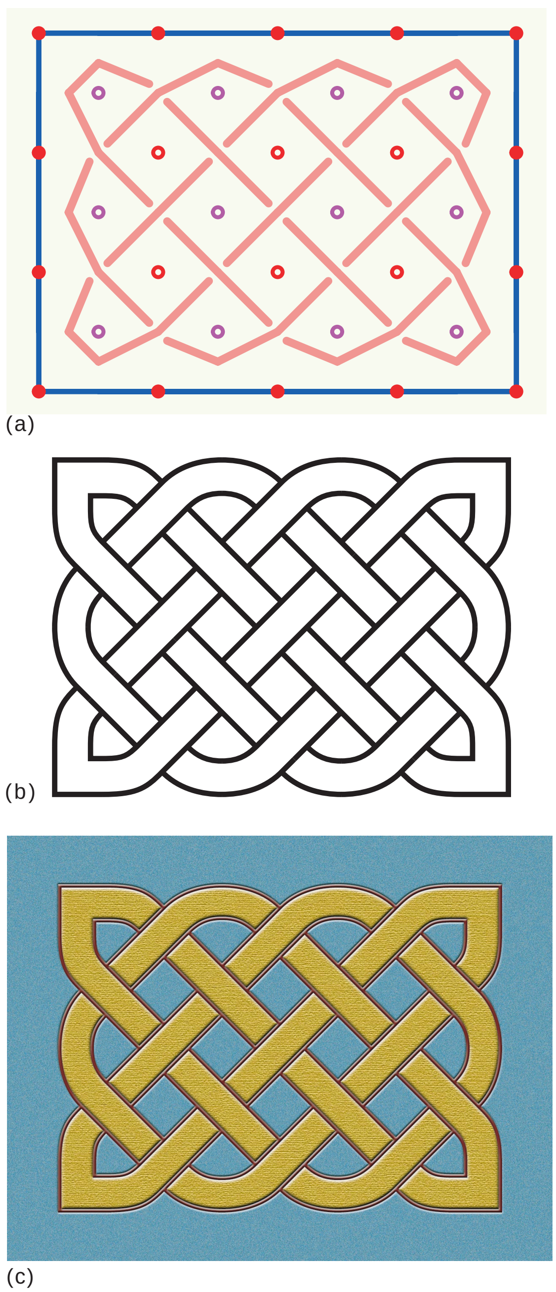
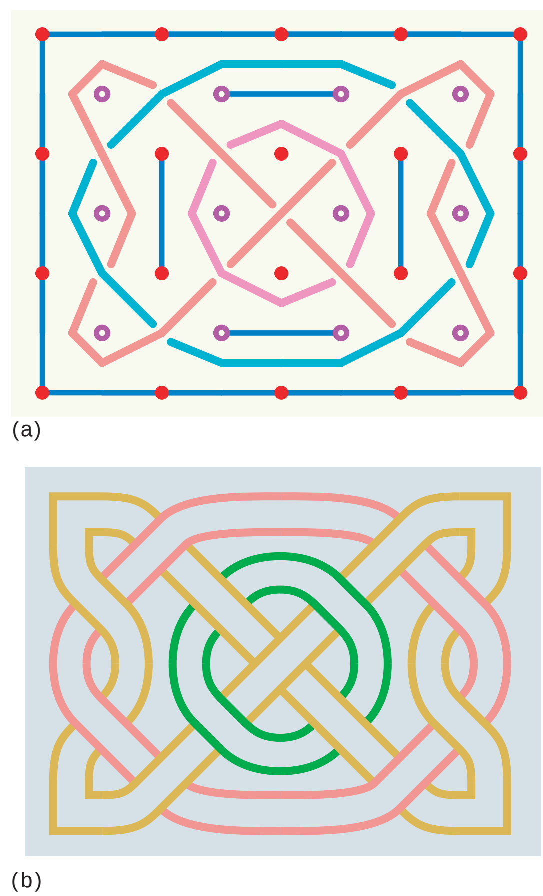
I first discovered Celtic knots almost 20 years ago. Since then, I've enjoyed working with them in a wide variety of art media. My passion for Celtic knots has even prompted me to write about them, which is one of the reasons why this column is special: it's part three of my first three-part column, my first column of the third millennium, and it's concerned with how to build 3D Celtic knots. I think there's a pattern here.

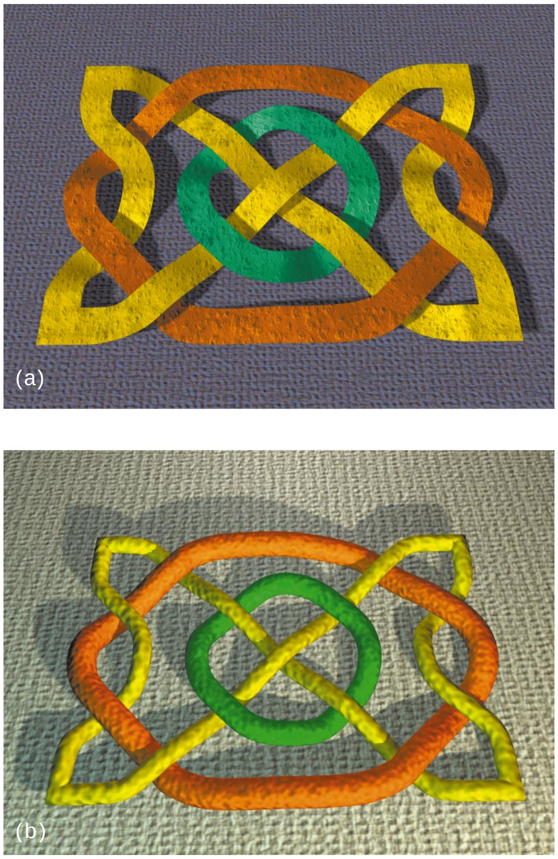
Let's recap quickly. Two issues ago I discussed my Celtic Knot Assistant program, which I wrote to help me create traditional 2D knotwork panels. The idea was to create several overlapping, rectangular grids of dots, place *breaklines* between neighboring dots, and then draw a continuous line called the *skeleton* from dot to dot, avoiding the breaklines. Figure 1a shows the idea. From the skeleton, you can create the *band*, which is like a wide ribbon running along the design. Figure 1b shows the band for Figure 1a. You can decorate the knot in a



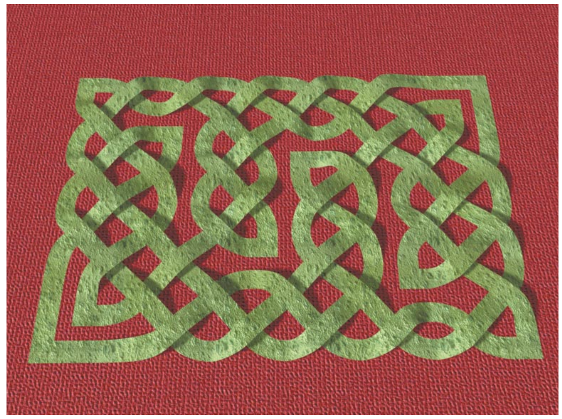
1 Recalling how to make a Celtic knot. (a) The red dots are the primary grid, and the purple dots are the secondary grid. The dark orange is the skeleton of the knot. (b) The resulting band. (c) A decorated version of the band.



2 (a) The skeleton for a three-ribbon knot. (b) The band for the knot.



3 **(a) The knot of Figure 2 represented as a 3D ribbon. (b) The same knot using tubes of different radii for the different bands.**



4 **A 3D knot based on a 9-by-7 grid pattern.**

wide variety of ways; Figure 1c shows one approach.

Most traditional designs are built from a single band, but multiple bands are also popular. Figure 2 shows a three-band knot, created by adding breaklines to the knot diagram of Figure 1a. The essential characteristics of Celtic knotwork are in the stylized decoration of the band, the choices of colors, and the requirement that all pieces of the band pass over and under each other in perfect alternation.

In the last issue I talked about replacing some of the overlap points with a different bit of geometry, so that the band pieces turn away from each other rather than crossing over. Carrying this to an extreme produces *snakes*, or knots that have been completely unraveled so that they don't overlap. I also talked about creating knots that aren't strictly based on square grids.

Out of flatland

Let's move into three dimensions. Perhaps the most straightforward leap is to take the implied over-and-under pattern of the knots in Figures 1 and 2 and actually create bands that rise up out of the plane for an overlap (or drop down for an underlap). Figure 3 shows two different ways to take the knot of Figure 2 into 3D.

Figure 4 shows another example of a single band knot.

I'll speak for a moment about programming this, in case you want to make images like these yourself. I had originally thought that adding 3D would be a pretty easy hack to my basically 2D Celtic Knot Assistant program. Wrong! There's nothing conceptually hard about it, but the mechanics of adapting my existing code proved tough, largely due to the many shortcuts I'd taken because I was only worried about 2D in the original program. The changes to the user interface were small. I added a setting for the height of the overlapping ribbon and a switch for whether the output should be a 2D Postscript file or a 3D Studio Max file.

For my 3D models, I placed most of the knot in a plane and lifted the band up out of that plane to create overlaps. The fun part of the process was finding a curve I liked for the band as it rose up and descended. I looked around for a nice blending curve, but wasn't able to find one that looked right for this job and had enough controls for me. So I cooked up a little curve of my own.

The recipe has a few steps. We start with a cubic curve $s(x)$ for $x = [0, 1]$. To give me a little more control, I first pass x through another function t that massages it by raising it to some power n. Because t only wants to work with values of x from 0 to 0.5, I start the process with a little splitting function u. Here are the details:

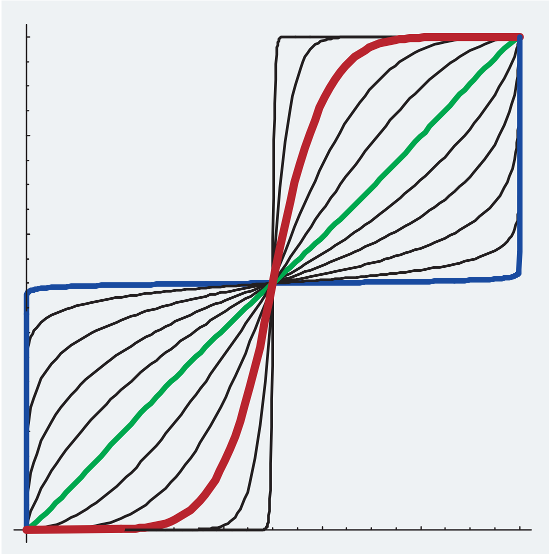
$$s(x) = -2x^3 + 3x^2$$

$$t(x, n) = s((2x)^n/2)$$

$$u(x,n) = \begin{cases} \text{If } x < 0.5 & t(x,n) \\ \text{Else} & 1 - t(1-x,\, n) \end{cases}$$

So you drive u with values of x from 0 to 1, and use the value of n to choose the kind of curve you want. Figure 5 (next page) shows what u looks like for different values of n. When $n = 0$, the function jumps to 0.5 almost immediately and stays there throughout the entire domain. When $n = \infty$, the function stays at 0 until $x = 0.5$, then jumps to 1. I like the fact that I can vary just one parameter to get from one extreme to the other, passing through a whole family of smooth blends along the way. I used $n = 3.5$ for Figures 3 and 4.

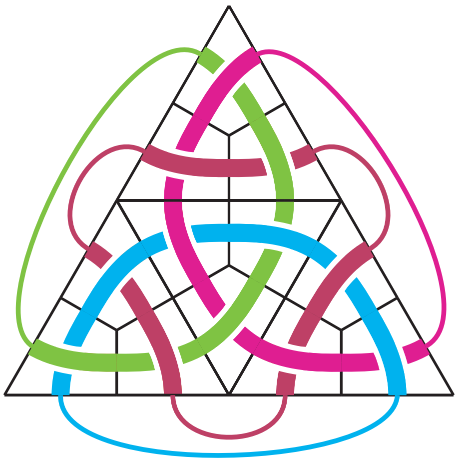
Turning the bands into tubes was straightforward. I wrote a file that contained a Bezier spline for the centerline of each band. Then manually in 3D Studio Max I used that as the path curve for an extruded spline, or lofted surface. This technique gave me an enormous



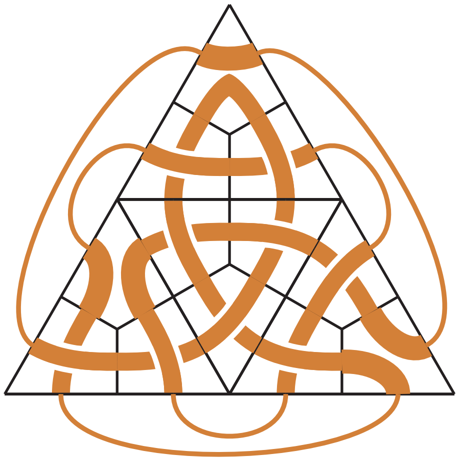
5 The function $u(x, n)$ plotted over $x = [0, 1]$ for different values of n. $n = 0.005$ is blue, then $n = 0.04, 0.11, 0.22, 0.37$. The curve $n = 0.6$ is green, then $n = 1, 1.8, 3.5, 10$, and finally $n = 100$. The thick red line represents the value I used for Figures 3 and 4, $n = 3.5$.

6 "Pisces," a two-band knot in 3D. This is part of a set of zodiac-themed knotwork. The yellow twisted rope isn't just a textured cylinder; it's actually a four-strand rope that was extruded along the fish curve and twisted along its length.

7 A Celtic knot built on a non-square arrangement of four-sided tiles.



(a)



(b)

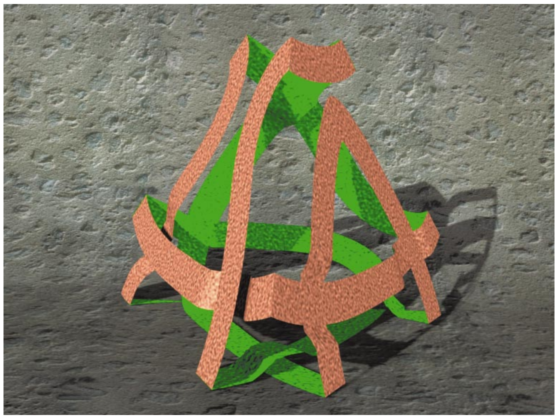
8 Building knotwork on a tetrahedron. (a) An unfolded tetrahedron, containing four triangles each made of three quadrilaterals. When folded up, this would result in four bands; the thin lines show how the pieces connect when the solid is folded up. (b) A modification of the network, so now we have only one band.

amount of design flexibility. Figure 6 shows an example of a two-band knotwork that I made this way.

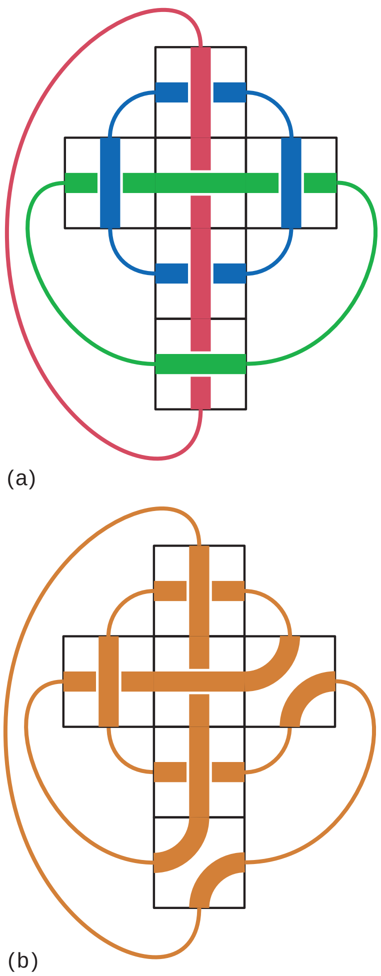
Getting off the grid

In part 2 I talked about how to create knotwork on grids that were more general than rectangular boxes. There's nothing that says you can only build knots on four-sided tiles, but I like them best because they let me work with two pieces of band in each tile. The tiles don't have to be square, or arranged in a grid, as Figure 7 demonstrates. It works best, though, if each tile has four sides.

Let's be a little more ambitious with our tiles. Figure 8a shows how we can take three four-sided tiles and arrange them into a triangle, then assemble four of these triangles into the net for an unfolded tetrahedron, or pyramid. I've colored the bands and included thinner lines outside the net to show where the edges join together when the pyramid is folded up. Think of the middle triangle as the base, and imagine pulling up its surrounding triangles by grabbing their outermost cor-



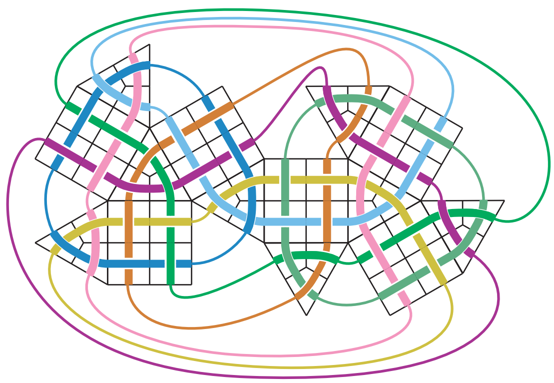
9 **The net of Figure 7b folded up into a tetrahedron. The one band is tan on the outside and green on the inside.**



10 **Celtic knots on the cube. (a) A standard knotwork pattern on the net of the unfolded cube. The thin lines show how the pieces join up when the cube is folded. This gives us three bands. (b) A modification to the knotwork resulting in just one band.**

ners and pulling them upward, so that all three corners meet. If you fold it up this way, the knotwork pattern in Figure 8a makes four bands.

11 **The cube of Figure 9b folded up into 3D, and then deformed to fit onto a sphere.**



12 **Knotwork on the net of an unfolded cuboctahedron.**

We saw last time that we can replace any tile containing an intersection with two pieces of band that avoid each other, joining up adjacent sides rather than opposite sides. In Figure 8b I've made a few of those replacements, and you can see that the result is now a single continuous band over the pyramid.

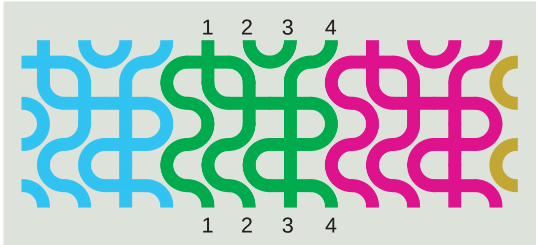
What does this really look like if it's folded up? Figure 9 shows a 3D version of Figure 8b.

If things worked nicely for the tetrahedron, I bet they'll look great on a cube. Figure 10a shows the net for an unfolded cube, and 10b shows the result when we reweave the band so it's only a single thread. Figure 11 shows the rendered, 3D version, where I've taken the liberty of projecting the basic plane of the band outward onto a sphere, so we have a Celtic knotwork ball.

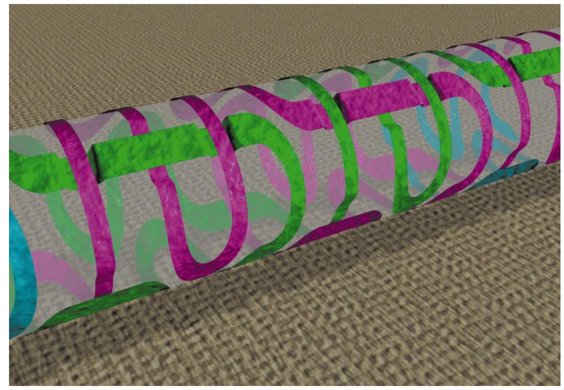
You can follow this as far as you'd like. Figure 12 shows an example for a cuboctahedron with normal weaving, creating seven bands. I'll leave it for you to modify some of the tiles so that this design forms a single continuous ribbon.

Celtic tubework

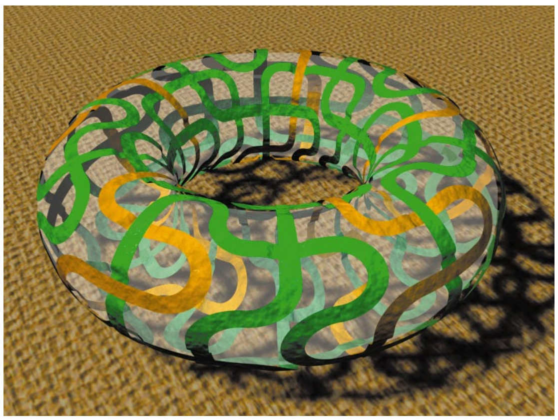
Let's take another approach, and create a knotwork cylinder. Figure 13 (next page) shows the net; the top edge matches up with the bottom edge when the tube is rolled up. In this particular weaving, we have a few



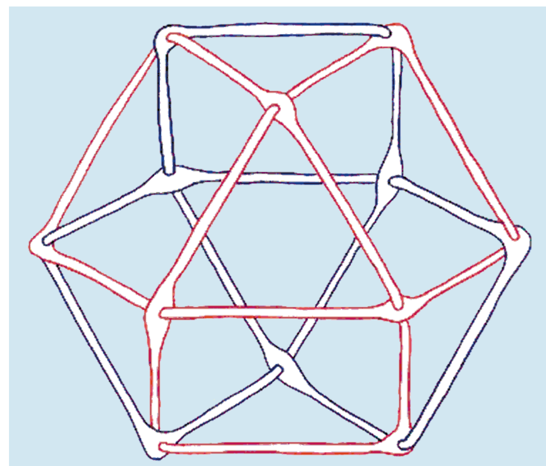
13 Building a knotwork tube. The top and bottom of the figure are joined up. For example, the numbers on the green band mark pieces that meet when the tube is formed.



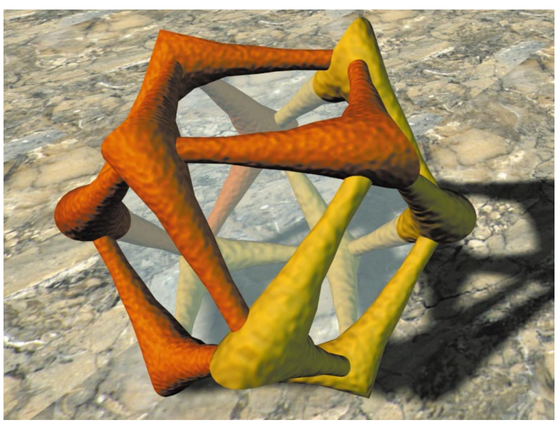
14 A knotwork tube (the band pattern is slightly different than the one in Figure 12).



15 A knotwork donut.



16 A hand-drawn tube that follows the edges of a cuboctahedron. There are two separate Celtic "ropes" that follow the same alternating pattern as a Celtic band in 2D.



17 Figure 15 rendered in 3D and seen from a slightly different point of view. If you follow the tubes around, you'll see that only two unique pieces exist.

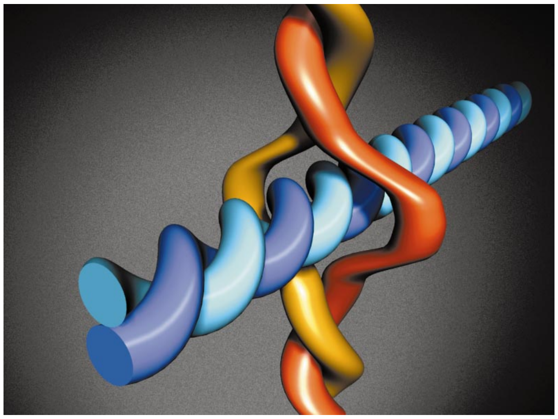
18 A Celtic icosadodecahedron.

different bands, each of which is rather large. Figure 14 shows the result in 3D, using a slightly different band pattern.

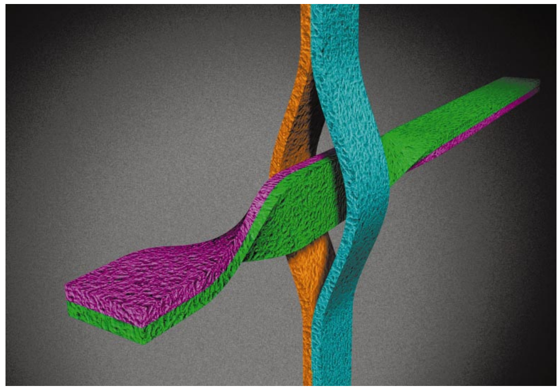
If we also match up the left and right sides of the unfolded net, we form a torus, or donut, as shown in Figure 15. Mmmm, knotwork donuts.

Another way to weave a band on a polyhedron is to pass it over the edges of the solid. Figure 16 shows a hand drawing of this idea on a cuboctahedron. Since we're now following the edges, I've drawn the knot pieces as tubes—I call this kind of design Celtic tubework. We have a lot of flexibility here in deciding how to represent the flat band's idea of "under" and "over." In this figure I chose to use the ideas of "through" and "around," so the tube passes through itself, then surrounds itself, in alternation. Figure 17 shows a rendered version.

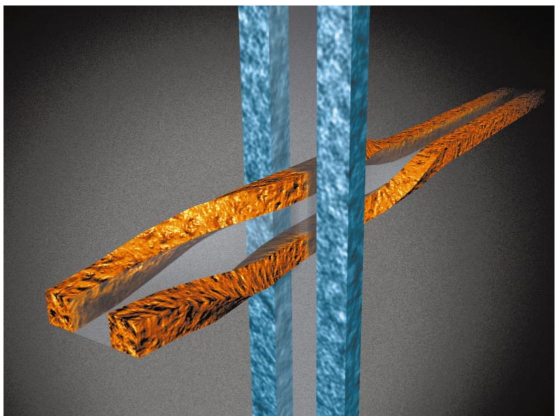
We can of course do the same thing on more complex figures. Figure 18 shows the result on a solid known as an icosadodecahedron, which is sort of a halfway blend between the two biggest Platonic solids—the 20-sided icosahedron and the 12-sided dodecahedron. Here I've kind of pushed the through/around idea a little farther



19 **Another way of building Celtic knots in 3D. The entwined pieces open up and pass through in alternation.**



20 **Another 3D variation. The parallel bands either pull apart or pass through one another in alternation.**



21 **Another 3D weaving method involving a twist of one band to pass through the other.**

22 **A one-band knotwork that is woven with the approach of Figure 21.**

and made the tube narrow down as though it was a chain of tapered dowels.

We can represent the over/under relationship in 3D in other ways. Figure 19 shows a way of treating the tube like a pair of intertwined cylinders. One pair unwinds and opens up to allow the other to pass through. Of course, on the next intersection the pair that opened up passes through a gap formed by a different segment. Figure 20 shows a treatment of the tube that pays some homage to the flat bands of knotwork. Two thick sheets that normally travel along stuck together represent the tube; one set opens up to allow the other to pass through when needed.

Figure 21 shows another variation on this twisting approach. Here two parallel rectangular tubes (I've bridged them here with a translucent section) represent the band. One band turns 90 degrees at the intersection to pass through the empty slot in the other. Figure 22 shows a one-band knotwork that uses this technique.

I love designs based on Celtic ideas, be it knotwork, tubework, or other variations. I love the variety of forms available even in such a seemingly simple technique. I've only begun to scratch the surface of the artistic and technical topics here. I plan sometime in the future to use Celtic knots as a springboard into the fascinating mathematical topic known as knot theory, and perhaps also return to some more interesting questions of computer-aided design and aesthetics. ■

Further Reading

These three columns only hint at the enormous variety of design possibilities that Celtic knotwork and knots in general can offer. Several readers wrote to me after the second installment of this trilogy pointing me to interesting Web sites. Robert Scharein developed an interactive program called KnotPlot for the SGI and PC platforms, which lets you build knots and render them. You can find it online at http://www.cs.ubc.ca/nest/imager/contributions/scharein/KnotPlot.html. Christian Mercat developed another approach to building Celtic knotwork that differs from the three-grid technique I used in my programs. His article is available in French at http://www.bok.net/kri/celte, and in English at http://www.abbott.demon.co.uk/mercatmethod.html. Steven Abbott implemented the technique in a PC-based design program, which can be downloaded from http://www.abbott.demon.co.uk/knots.html.

If you're interested in knot theory in general, a terrific introductory book is *The Knot Book* by Colin C. Adams (W.H. Freeman and Sons, New York, 1994). If you'd like to delve a little deeper into the mathematics, I suggest *On Knots* by Louis H. Kauffman (Princeton University Press, Princeton, N.J., 1987).

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