

## Venn and Now

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"I know what you're thinking about," said Tweedledum, "but it isn't so, nohow."  
"Contrariwise," continued Tweedledee, "if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic."  
—*Through the Looking-Glass*, Lewis Carroll

**W**ho knows what the truth really is? Not just temporary or relative truth, but eternal and universal truth? Truth with a capital T.

Nobody knows the Truth, of course. However, that hasn't stopped people from looking. One of the results of that search has been systems of formal logic that try to turn the work of reasoning and arguing into mechanical processes, where errors of interpretation and execution can be automatically detected and corrected. Formal logic can't tell us much about the human condition, but we can use it to reason our way through the world.

One of the most famous visual tools for logic is the *Venn diagram*, which most frequently shows the relationship between three object classes by drawing three overlapping circles.

In this article I'll talk about Venn diagrams, how we can generalize them to more than three sets of objects, and some related tools for visual logic.

### Logical arguments

The goal of all the visual techniques we'll cover in this article is to help us decide on a class of logical questions. To get the ball rolling, I'll start by defining some useful terms.

A *logical argument* is a sequence of statements. We call the last statement the conclusion; the others are called variously the premises, hypotheses, or assumptions. Each statement, including the conclusion, refers

to the existence of one or more object classes. A logical argument is valid if the conclusion is always true when all of the assumptions are true. Any logical argument that isn't valid is invalid.

Here's an example of a logical argument with two premises and a conclusion:

*Premise 1:* It rains a lot in the Northwest.

*Premise 2:* Seattle is in the Northwest.

*Conclusion:* It rains a lot in Seattle.

In this case, the argument is valid. How about this one?

*Premise 1:* Some dogs are known for their bark.

*Premise 2:* Some trees are known for their bark.

*Conclusion:* Some dogs are trees.

This argument is obviously invalid.

Let's look first at the structure of valid statements, and then at how we use them in patterns to create arguments.

Statements typically use a small vocabulary of common terms. The connective terms *and*, *or*, and *not* have their common interpretations.

We can use the if/then construction to build a logical chain of true statements. For example, I might say, "If I'm driving my car then I'm sitting down." Let's call this statement *S*. Because an if/then construction has two conditions, and each condition can be true or false, four possible combinations exist for the entire statement. Let's rephrase *S* abstractly as "if *a* then *b*." Let's now make four observations, one for each combination of true and false for *a* and *b*. We'll say that *S* is true or false depending on whether it matches our observation. Table 1 shows the four choices and the results.

The first two of these decisions are reasonable given how we normally use language. Cases 3 and 4 are perhaps unusual. That's because the rule for interpreting a formal if/then statement is that if the first clause isn't satisfied then the whole statement is true. The princi-

Table 1. Checking the validity of the statement *S* "if I'm driving my car then I'm sitting down" (where *a* = "I'm driving my car" and *b* = "I'm sitting down").

Case	<i>a</i>	<i>b</i>	Observation	Statement <i>S</i>
1	True	True	I'm driving my car, and I'm sitting down.	True
2	True	False	I'm driving my car, and I'm not sitting down.	False
3	False	True	I'm not driving my car, and I'm sitting down.	True
4	False	False	I'm not driving my car, and I'm not sitting down.	True

ple here is that if the first clause isn't met then the whole thing is irrelevant. Although the statement is irrelevant, it's certainly not false. And whatever isn't false, is true.

So in any if/then statement, if the clause isn't satisfied then the whole statement is taken to be true. For example, let's suppose that there are no talking dogs. Then the statements "all talking dogs enjoy Shakespeare," "some talking dogs like to water ski," and "no talking dogs enjoy sipping coffee in outdoor Parisian bistros" are all true, because the first clause is false.

One of the most common errors in logical reasoning is getting the chain of cause and effect wrong. The opposite of "if  $a$  then  $b$ " isn't "if  $a$  is false then  $b$  is false." Let's try that out with the language of our example  $S$ : "if I'm not driving my car then I'm not sitting down." That's clearly not going to be true in general. In fact, I'm sitting down right now as I type this article, but I'm not in my car. We call the correct logical opposite the *contrapositive*, and we express it as "if  $b$  is false then  $a$  is false." Note that  $b$  comes first. In fact, this is exactly the same as the original statement  $S$ . The contrapositive of  $S$  is, "if I'm not sitting down then I'm not driving my car." Now that makes sense.

A phrasing more common to our everyday use of if and then is the *if and only if/then* statement. For example, "if and only if it's cold outside then I'll turn on the heat." In this case, we know that if the heat is on then it's necessarily cold outside and vice versa.

It's easy to mix up the rules for *if/then* and *if and only if/then*, but the rules of logic are careful about keeping them distinct.

It's also useful to be able to make statements about whole classes of variables.

The phrases *for any*, *for all*, and *for every*, are synonyms that let us make a claim about every element in a class. For example, we could say, "For all birds, birds have wings." Often such statements are compressed a bit, as in "all birds have wings." We can also phrase this in an equivalent form that expresses the nonexistence of a class: "no birds are wingless."

If we want to assert that at least one element of the class has some property, we use phrases like *there exists*, *there's at least one*, and *there is an*. These are all synonyms that let us claim that at least one thing with this property exists. For example, we might say, "There exists a bird that can't fly." This doesn't mean that all birds can't fly, or that there are even two. It just means that there's at least one.

The phrases *for all* (and its variants) and *there exists* (and its variants) are called *quantifiers*.

### Prove it ain't so

We can usually determine if an argument is valid or invalid using only three basic tools.

First, we can simplify the statements using *tautologies*, or known true statements. Tautologies are useful because they typically express true statements in slightly different terms than those used in the original statements. A tautology adds no new information to an argument, but helps us convert or simplify it.

The second tool for proving an argument is to search for a contradiction. If we assume the premises are true,

### Standard Notations

$\forall a$	For all $a$
$\exists a$	There exists an $a$
$a \wedge b, ab$	$a$ and $b$
$a \vee b, a + b$	$a$ or $b$
$\sim a, a', \bar{a}$	Not $a$
$a \rightarrow b$	If $a$ then $b$

a contradiction leads to a conclusion that's the opposite of the conclusion in the argument. Sometimes we can approach this from within the argument, and sometimes we find it useful to apply tautologies to transform one or more of the statements into a more convenient form.

The third tool, and the one I'll focus on in this article, is to convert everything into symbols and then manipulate those symbols in a mechanical way, without concerning ourselves with what they stand for. In this symbolic form, we often use the notation mentioned in the sidebar "Standard Notations." For example,  $ab$  means " $a$  and  $b$ ," or more completely, "both  $a$  and  $b$  are true." The expression  $ad'$  means " $a$  and not- $d$ ," or " $a$  is true and  $d$  is false."

Now that we know the statements' forms, let's look at what happens when we bring them together to form arguments.

The first person to study chains of symbolic statements was Aristotle, who gave the name *syllogisms* to patterns of arguments. He showed how to apply inference rules to arguments to determine their validity, using a technique called deductive reasoning.

Aristotle's insight was that if we can convert our statements into symbolic form then we can apply standard patterns to those symbols. We similarly use symbols to solve word problems. For example, we might be asked, "Bob has three chickens and bought four more chickens. How many chickens does he have?" We convert this word problem into the symbolic expression  $3 + 4 = ?$ , and apply our rules of arithmetic to find the answer. Then we convert the symbolic result, 7, back into the vocabulary of the original problem, leading to our conclusion (Bob has too many chickens).

Here are four of Aristotle's patterns with examples (recall that  $a \rightarrow b$  means "if  $a$  then  $b$ "):

*Modus Ponens* (method of affirming):

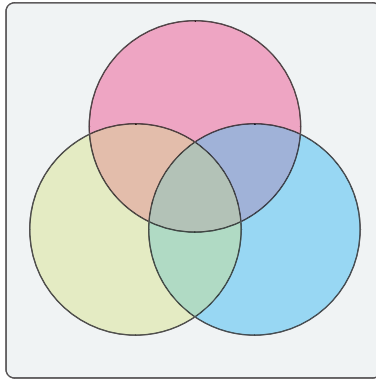
1.  $a \rightarrow b$ : If it's a folk song, it has just three chords.
2.  $a$ : It's a folk song.
3. Therefore,  $b$ : It has just three chords.

*Modus Tollens* (method of denying):

1.  $a \rightarrow b$ : If the shoe fits, I'll wear it.
2.  $b'$ : I'm not wearing the shoe.
3. Therefore,  $a'$ : The shoe doesn't fit.

*Disjunctive syllogism*:

1.  $a \vee b$ : My master plan is either brilliant or insane.
2.  $a'$ : My master plan isn't brilliant.
3. Therefore,  $b$ : My master plan is insane.



**1** Three circles of the traditional three-element Venn diagram.

*Hypothetical syllogism:*

1.  $a \rightarrow b$ : If it's on the Web then it's true.
2.  $b \rightarrow c$ : If it's true then it's important.
3. Therefore,  $a \rightarrow c$ : If it's on the Web then it's important.

I should note here that three important oppositions arise when working with logical arguments: the *contradiction*, *contrapositive*, and *converse*.

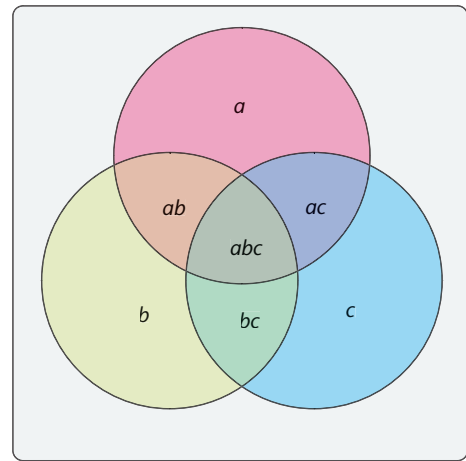
To prove a contradiction we just apply these rules to the premises to arrive at a conclusion that's incompatible with the stated conclusion. This proves that the argument is invalid.

If we're given a statement "if  $a$  then  $b$ " (or  $a \rightarrow b$ ) then the contrapositive is "if  $b'$  then  $a'$ " (or  $b' \rightarrow a'$ ). If we can prove the conclusion's contrapositive then we also prove the conclusion itself.

The converse is an error of reasoning that goes like this:

1.  $a \rightarrow b$ : If it's chocolate, it's delicious.
2.  $b$ : It's delicious.
3. Therefore,  $a$ : It's chocolate.

This conclusion is wrong. So just because the conclusion of an if/then statement is true, that doesn't mean that the premise is true; the conclusion could be true for other reasons.



**2** Venn diagram of Figure 1 with labels for each region. The white unlabeled region outside the circles represents elements that aren't within any sets.

Now that we have all this terminology under our belts, let's look at representing logical arguments visually.

**Three-Venn diagrams**

Figure 1 shows the famous Venn diagram for three object classes, sometimes called a *three-Venn diagram*. If we name these three classes  $a$ ,  $b$ , and  $c$  then we can label each of the diagram's regions as in Figure 2. Note that there are eight regions, corresponding to these three classes' eight possible combinations. Suppose that we assign the three classes this way:

1.  $a$  = things that are scary movie monsters
2.  $b$  = things that are unusually large
3.  $c$  = things that are airborne

Table 2 shows examples of the eight combinations. Remember that if a category is negated, that means the object doesn't have that trait.

I placed these in the appropriate zones in Figure 3. Note that the white outer region contains all those things that don't belong within one or more circles.

This is an example of using Venn diagrams for *classifying*. Here we have three criteria, and we place objects

in the diagram according to which of these criteria they satisfy. You can pick any three traits, plug them in for  $a$ ,  $b$ , and  $c$ , and then populate all eight cells (don't forget the outer cell corresponding to  $a' b' c'$ ).

We can also use Venn diagrams to follow a chain of statements and see if they form a valid conclusion. For example, consider the following logical argument:

1. All winged monkeys have tails.
2. All doctors are tailless.
3. Therefore, no winged monkeys are doctors.

Table 2. Examples of the eight possible combinations of classes  $a$ ,  $b$ , and  $c$ .

Scary Movie Monsters	Unusually Large Things	Airborne Things	Example
$a'$	$b'$	$c'$	A puppy
$a'$	$b'$	$c$	A helium balloon
$a'$	$b$	$c'$	The pyramids
$a'$	$b$	$c$	A pterodactyl
$a$	$b'$	$c'$	The Blob
$a$	$b'$	$c$	Dracula
$a$	$b$	$c'$	Godzilla
$a$	$b$	$c$	Giant atomic grasshoppers

Is this valid? The first thing we do is convert it into symbolic form:

1.  $a$  = things that have tails
2.  $b$  = winged monkeys
3.  $c$  = doctors

With these substitutions, we get

1. All  $b$  are  $a$ .
2. All  $c$  are  $a'$ .
3. Therefore, no  $b$  are  $c$ .

Is this true? Figure 4 shows the steps. In Figure 4a I marked in red (meaning empty) the two cells  $a'bc$  and  $a'bc'$ . This is because “all  $b$  are  $a$ ” is equivalent to “no  $b$  are  $a'$ .” Similarly, in Figure 4b I marked in red cells  $abc$  and  $a'b'c$ . I could mark these cells as empty because I converted “all  $c$  are  $a'$ ” to the equivalent “no  $c$  are  $a$ .” Figure 4c puts the diagrams together, and to confirm that “no  $b$  are  $c$ ,” we need only look to see if any  $b$  cells that overlap with  $c$  aren't empty. They don't, so the logical argument is true.

Let's look at a second example with four statements:

1. Some children are intelligent.
2. Some government officials are intelligent.
3. No government officials are children.
4. Therefore, some government officials are intelligent adults.

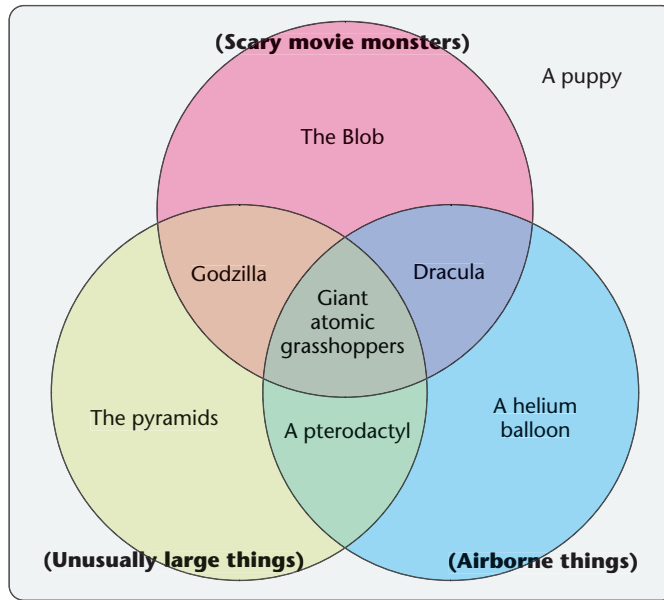
As before, first we convert this to symbols:

1.  $a$  = children
2.  $b$  = government officials
3.  $c$  = intelligent people

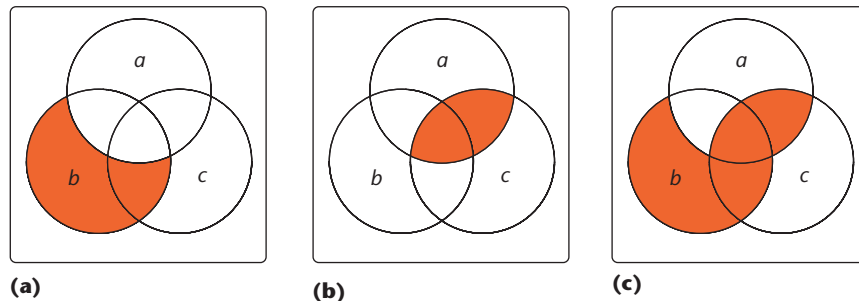
The argument's symbolic form is

1. Some  $a$  are  $c$ .
2. Some  $b$  are  $c$ .
3. No  $b$  are  $a$ .
4. Therefore, some  $b$  are  $ca'$ .

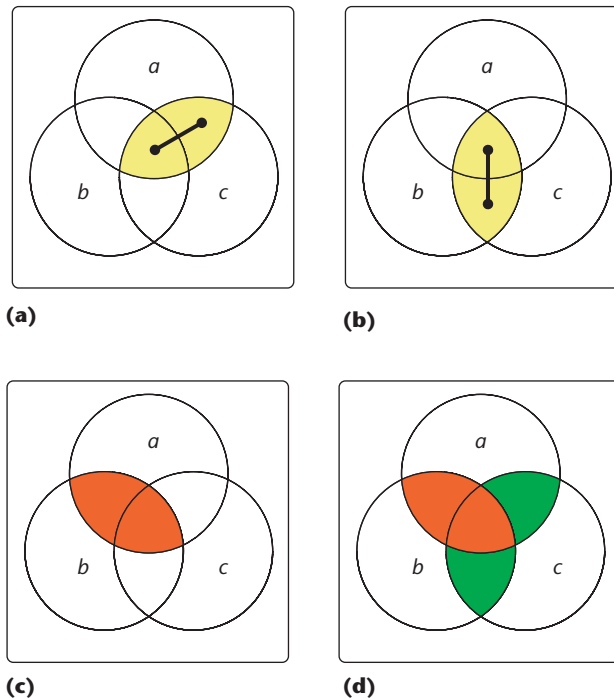
Figure 5a shows the first statement, “some  $a$  are  $c$ .” Note that I marked in yellow the two cells  $ac$  and  $abc$ , and joined them with a connective line. This means that we know that at least one of them is occupied, but not necessarily both. Similarly, I marked the cells  $bc$  and  $abc$  in Figure 5b the same way. Figure 5c resolves



3 Venn diagram of Figure 2 with examples placed in the cells.



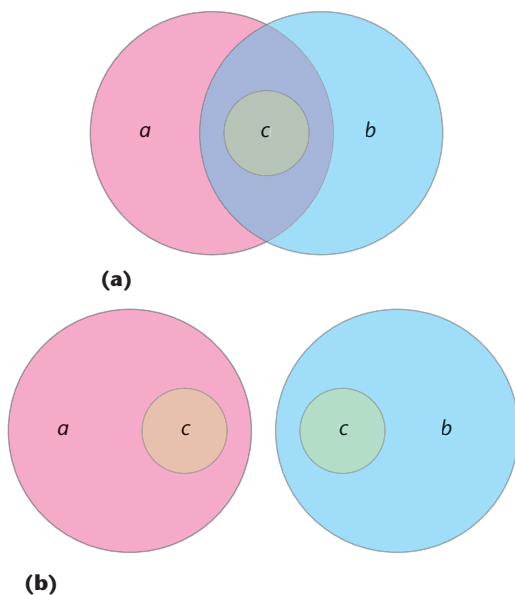
4 Worked-out syllogism with the three-circle Venn diagram. Regions marked in red are empty. The premises for each are (a) all  $b$  are  $a$ , (b) all  $c$  are  $a'$ , and the conclusion is (c) no  $b$  are  $c$ .



5 Worked-out four-statement syllogism with the three-circle Venn diagram. The premises for each are (a) some  $a$  are  $c$ , (b) some  $b$  are  $c$ , and (c) no  $b$  are  $a$ . The conclusion is (d) some  $b$  are  $ca'$ .

6

Inconsistencies in Venn diagrams. (a) An incorrect diagram that might seem to result from "all  $c$  are  $a$ " and "all  $c$  are  $b$ ." (b) A correct (but temporary) diagram.



the issue for us by turning off cell  $abc$ . I'll use green to indicate the cells that are certainly occupied. Figure 5d shows that both  $ac$  and  $bc$  must therefore be occupied, while  $abc$  is empty. Thus, the conclusion that "some  $b$  are  $ca$ " is also true, because we have occupancy in cell  $bc$  but none in  $abc$ . Therefore, the argument is valid.

**How much of nothing do you have?**

There's an interesting subtlety that we need to address. The Venn diagrams with three overlapping circles aren't the only way to draw such things. The idea is simply to graphically capture the relationships between collections of objects, with shared characteristics represented by overlapping sets. We'll find this more general approach useful for our discussion here.

Suppose that we have two premises: "all  $c$  are  $a$ " and "all  $c$  are  $b$ ." This seems to suggest a Venn diagram like Figure 6a, where  $c$  is completely within  $a$ , and also within  $b$ . Two conclusions we can draw from this diagram are "some  $b$  are  $a$ " and "some  $a$  are  $b$ ."

That's not quite right, though. Suppose that no elements of the class  $c$  exist. Then the two premises are both trivially true. After all, if there are no elements  $c$  then anything we assert about them is true. If that's the case, though, then the overlap of  $a$  and  $b$  in Figure 6a is misleading.

To make this concrete, let's assign class  $a$  to intelligent things,  $b$  to scissors, and  $c$  to extraterrestrials. Let's assume for the moment that no extraterrestrials exist. Then I can assert both "all extraterrestrials are intelligent things" and "all extraterrestrials are scissors." Remember that since we've assumed that there are no extraterrestrials, both of these statements are defined to be true according to the rules of if/then statements that we saw earlier. From these premises we could conclude that "some scissors are intelligent things," which most people would agree isn't true.

Lewis Carroll was aware of the subtleties that occur when things don't exist. Here's a scene that appears in

the Mad Hatter's tea party from *Alice's Adventures in Wonderland* (1865):

"Take some more tea," the March Hare said to Alice, very earnestly.  
 "I've had nothing yet," Alice replied in an offended tone, "so I can't take more."  
 "You mean you can't take less," said the Hatter, "it's very easy to take more than nothing."

You can try to define your way around the problem of not knowing what to do with nothing, and Carroll did so in his books. He defined a proposition "some  $a$  are  $b$ " to mean three things: "some  $a$  exist," "some  $b$  exist," and "some  $a$  are  $b$ ." This approach has its own drawbacks and hasn't been widely adopted.

One way to handle this situation is to use a diagram like Figure 6b. Here we reserve judgment about the overlap between  $a$  and  $b$  until we receive more information. This diagram can be as misleading as Figure 6a, because it seems to imply "no  $a$  are  $b$ ." We should treat this as a provisional, or working, diagram, pending more information that may yet bring regions  $a$  and  $b$  together.

Although explicitly handling the status of undefined things is important, it also makes everything a little messier. For this article, I'm going to adopt Carroll's approach. If I make a statement like "some apples are fruit," I'll mean it to imply that there do exist things that are apples, and things that are fruit.

**Four-Venn diagrams**

What if we want to show examples of objects that share four traits? Or what if we want to prove the truth of a logical argument with four variables? The obvious answer is to use a *four-Venn diagram*, the generalization of the three-Venn diagram.

Figure 7 shows a reasonable first step, where I took four circles and overlapped them. Is this actually a four-Venn diagram? One way to check quickly is to see if the diagram has the necessary number of cells. But how many cells should we have?

The easy way to answer this is to think of each cell in a diagram with  $n$  traits as an  $n$ -bit binary number. For example, 0110 would indicate a cell where the second and third traits were included, but the first and fourth weren't. That is, it's an overlap between just the second and third regions. For a diagram with  $n$  regions, we'll have one cell for every binary combination from 0 to  $2^n - 1$ , or  $2^n$  cells in all. When  $n = 4$ , we should have 16 cells. A quick check of Figure 7 turns up only 14.

What's wrong? We're missing two cells:  $ac$  and  $bd$ . I monkeyed around with this diagram quite a bit, and I managed to get one of the missing cells represented. However, I couldn't figure out how to create a nice Venn diagram on the plane just by distorting the circles.

It's probably worth noting what nice means in this case. If you look at Figure 1 you can see a couple of characteristics I'd like to preserve. First, each cell, including the outermost cell, appears once and only once. Second, the cell shapes are simple. That is, they're not long serpentine shapes that twist around. Finally, the cells are arranged in a way that matches a



gray code. This is a way of counting in binary such that each number differs from the previous one in only one bit. For example, to count in a 3-bit gray code we'd write 000, 001, 011, 010, 110, 111, 101, 100. If you look at any two adjacent cells in Figure 2 you'll notice that they have this property, so when we cross a border, we gain or lose a single element: never more, never less. This creates a nice visual cohesion in the diagram, so that we can roam around it with our eyes and see gradual changes. It also tells us that objects next to each other in a classification diagram are similar, differing only in exactly one trait.

How might we capture these criteria in a four-element Venn diagram?

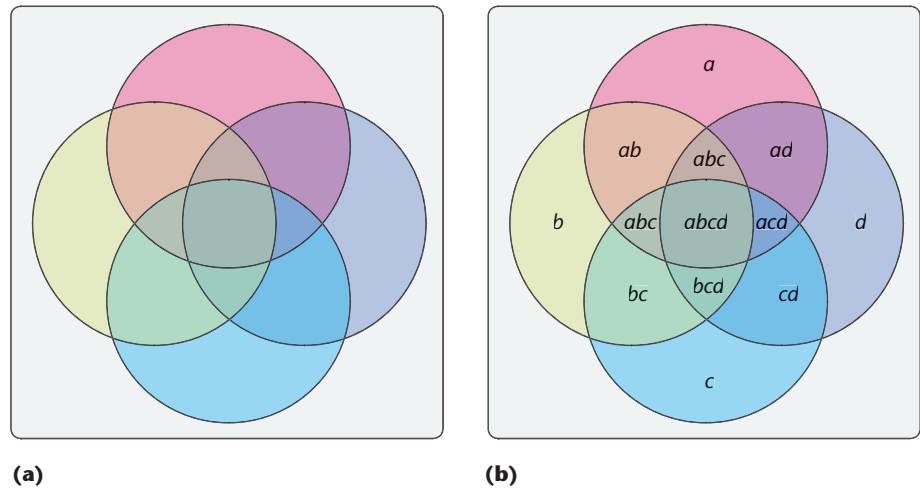
Figure 8 shows another way to construct the three-cell diagram, and Figure 9 shows the result of applying this approach to four cells. Unfortunately Figure 9 is still missing two cells. The lesson I learned from this is that simply jiggling around the cell shapes isn't going to get us to a four-Venn diagram. We need a new approach.

In this article I've included some quotes from Lewis Carroll's books. I used them not only because they're clever and illuminating, but because Carroll also worked on this problem of multicell diagrams. In his book *Symbolic Logic*, published in 1896, he presented what he called the *bilateral diagram* for solving logical arguments involving two variables. Naturally enough, he used the *trilateral diagram* for arguments in three variables. Figure 10a (next page) shows a version of his trilateral diagram and Figure 10b marks the cells. Figure 11 shows the region associated with each variable.

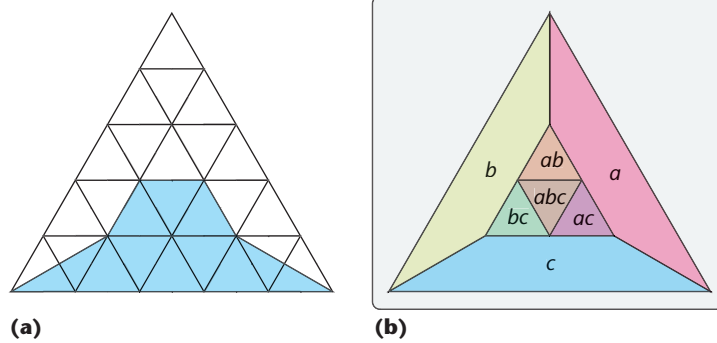
Note that Carroll started with a different approach than that used in the Venn diagrams. Rather than drawing regions within an outer cell, he takes the whole world (the surrounding rectangle) and cuts it in half vertically and horizontally, creating four cells.

Let's work a syllogism with the trilateral diagram:

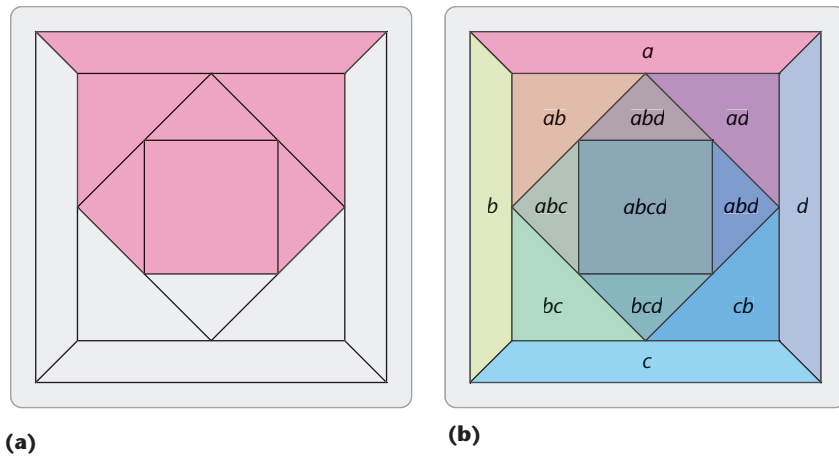
1. No pedestrians are pilots.
2. Some babies are pedestrians.
3. No drivers are airplane passengers.
4. Therefore, some drivers are pilots.



7 Attempt to create a Venn diagram with four circles. (a) Regions and (b) regions with labels. Note that regions *ac* and *bd* are missing.



8 Alternative way to draw the three-element circle diagram of Figure 1.



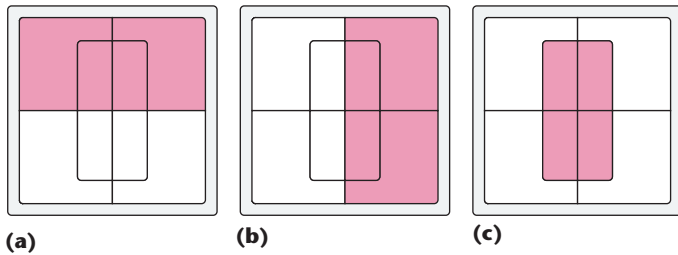
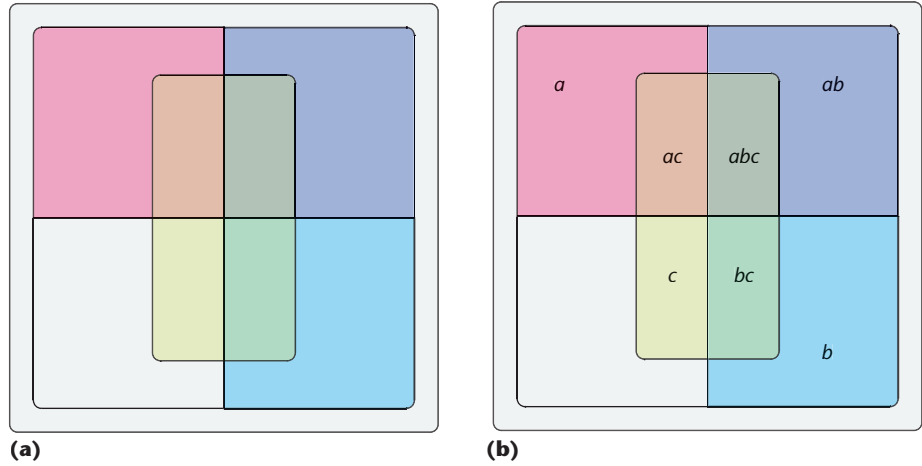
9 Alternative way to draw the four-element circle diagram of Figure 7.

Making the symbolic substitutions

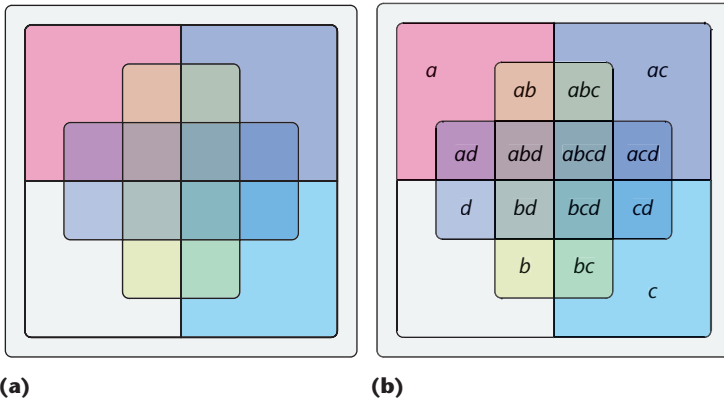
1.  $a$  = babies,
2.  $b$  = pedestrians ( $b'$  = drivers), and
3.  $c$  = pilots ( $c'$  = airplane passengers),

we can write the argument symbolically as

**10** Lewis Carroll's three-element trilateral Venn diagram showing (a) regions and (b) labels.



**11** Regions of the trilateral diagram. (a) Region  $a$ , (b) region  $b$ , and (c) region  $c$ .

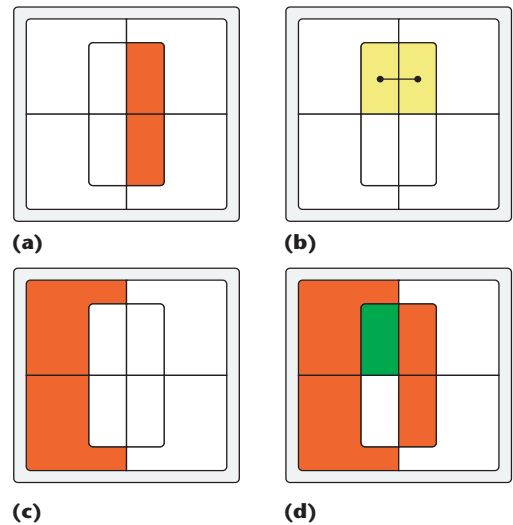


**13** Lewis Carroll's four-element quadrilateral diagram showing (a) regions and (b) labels.

1. No  $b$  are  $c$ .
2. Some  $a$  are  $c$ .
3. No  $b'$  are  $c'$ .
4. Therefore, some  $b'$  are  $c$ .

Figure 12 works through this argument, and you can see from Figure 12d that the conclusion is true, so the argument is valid.

The next step was his *quadrilateral diagram*, shown in Figure 13. To add a fourth class of objects, Carroll drew a new rectangle. Note that the new cell boundary passes through each of the existing cells once; we'll see



**12** An argument worked out on the trilateral diagram. (a) No  $b$  are  $c$ , (b) some  $a$  are  $c$ , and (c) no  $b'$  are  $c'$ . (d) Putting it all together, we can verify the conclusion that some  $b'$  are  $c$ .

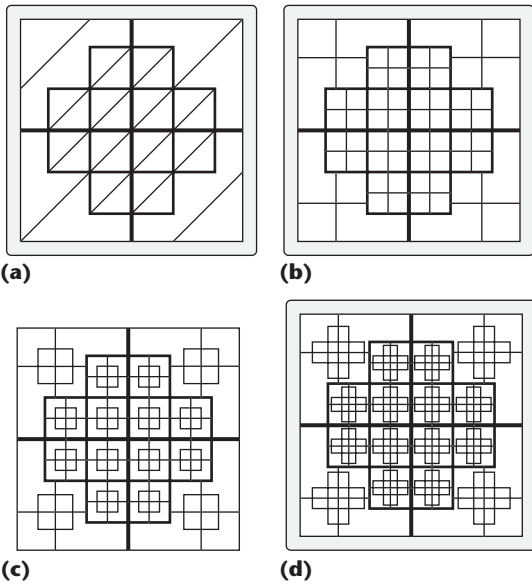
that this is a general rule for adding new traits to an existing diagram.

**Five and more**

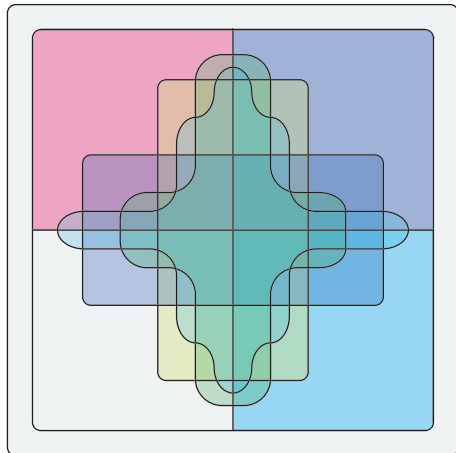
We've successfully managed to create a diagram for four variables, so let's try for five.

You won't be surprised to learn that Carroll had a go at this problem. His solution is Figure 14a, where he just cut each cell in two with a diagonal line. However, I found the result disappointing. It's not very nice to look at, but more importantly, it doesn't have the gray code property. To my mind, this makes it almost useless as a visual tool for anything but the most mechanical of proof checking. Carroll continued extending his diagrams in the most straightforward way, as Figures 14b through 14d show.

There's a nice solution right around the corner, though. I was looking at Carroll's quadrilateral diagram and wondered if I could cut through every cell with a single boundary, as he did to create the quadri-



14 Carroll diagrams for many variables. (a) Five, (b) six, (c) seven, and (d) eight elements.



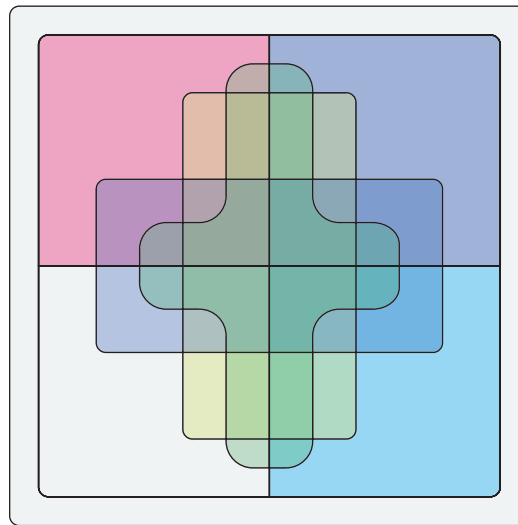
16 My six-element Venn diagram based on a square design.

lateral diagram from the trilateral one.

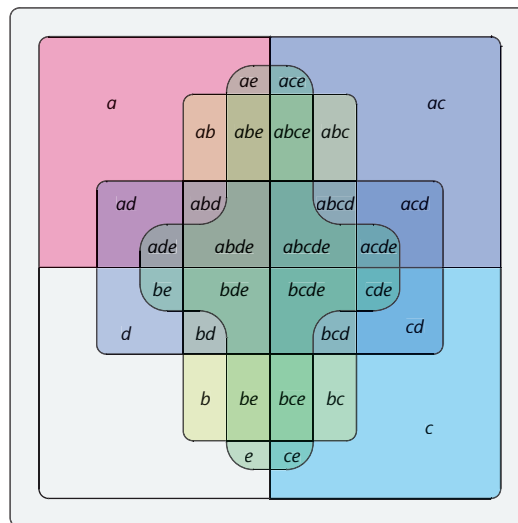
After a little fooling around with pencil and paper I came up with the solution of Figure 15. That seems to satisfy all the criteria, and it's not bad looking. If you check, you'll see that all  $2^5 = 32$  cells are present and appear only once, and that the gray code is satisfied: each cell boundary changes only one variable.

That got me going. I had to try my hand at six variables, with the 64-cell result of Figure 16.

I invite you to cook up some logical arguments in five or six variables and have a go at proving a conclusion with these diagrams. It's kind of fun to set up the argument symbolically (that is, "all  $b$  are  $d$ ") and then after you have a bunch of statements, assign properties to them (such as  $b$  = insects and  $c$  = carnival attractions). As long as you create statements that originally make some degree of



(a)



(b)

15 My five-element Venn diagram based on a square design showing (a) regions and (b) labels.

sense, you can discover some pretty goofy truths.

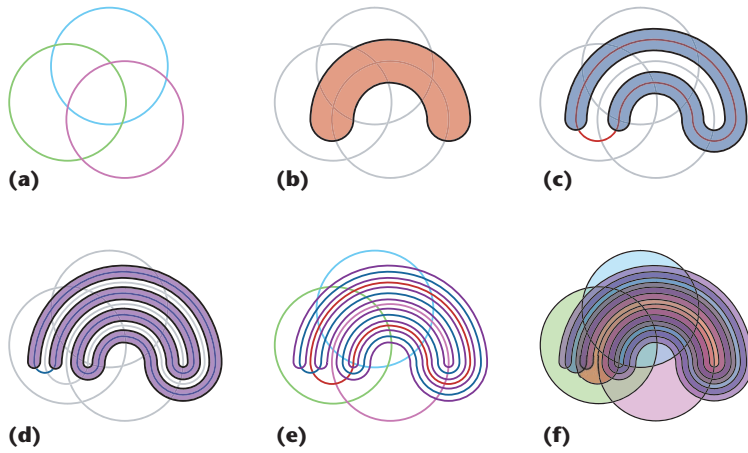
### Other solutions

You can find other ways to cook up interesting Venn diagrams. Venn himself suggested a general construction technique based on his three-circle technique. Figure 17 (next page) shows the steps. We begin with the three circles in Figure 17a and add a thick half-circle in Figure 17b. Recall that the general trick is to draw a line that crosses all the existing cells, so the simplest approach to extend this is to follow this half-circle, as in Figure 17c. You can keep going to Figure 17d and beyond, as long as your patience holds and you can draw the tiny lines between the curves!

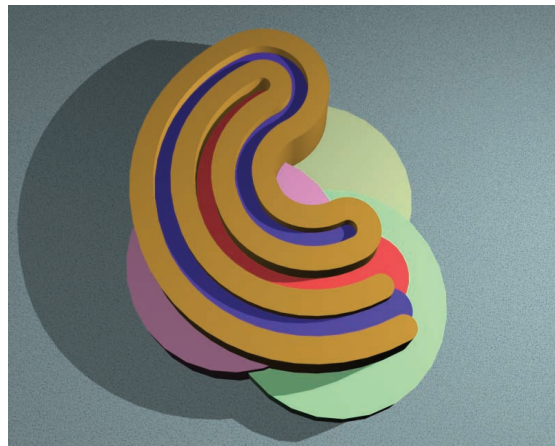
Because Figure 17f is hard to interpret, I created a 3D version with an extruded view (Figure 18).

This isn't a terribly nice solution. Could there be other approaches that retain the nice circular forms of the simple three-Venn diagram?





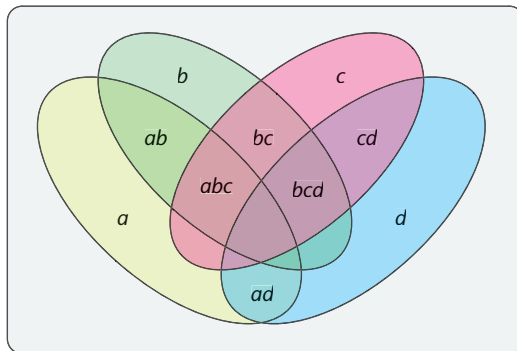
**17** Venn's construction for increasing the number of elements beyond three, based on the original three-circle diagram.



**18** A 3D version of Figure 17f.

In Carroll's *Symbolic Logic*, he presents a four-ellipse solution by Venn, as Figure 19 shows.

More recently, Branko Grünbaum found a lovely arrangement of ellipses for a five-element diagram, shown in Figure 20. This approach doesn't seem to generalize; I don't know of any Venn diagrams of six or



**19** A four-element diagram made out of four congruent ellipses.

more congruent ellipses.

Anthony Edwards developed another approach recently. The first two steps are like those of Carroll's: take the plane and cut it in half vertically and then horizontally, as in Figures 21a and 21b. However, where Carroll places a square, Edwards places a circle (see Figure 21c). Now draw a big serpentine curve that cuts all the cells, as in Figure 21d. To keep going, draw ever-denser serpentine curves that follow the circle, as in Figures 21e through 21i.

This figure meets all of our desires for a nice diagram. It's also easy to program and extend to any number of terms. Unlike Carroll's diagrams, it continues to look good even for high numbers of terms.

Let's take a look at proving a logical argument using a six-term Edwards diagram:

1. No  $b'$  are  $d'$ .
2. No  $g$  are  $d'f$ .
3. No  $d$  are  $f'$ .
4. Some  $b$  are  $ed$ .
5. No  $a'$  are  $dg$ .
6. Therefore, no  $a'$  are  $b'cfg$ .

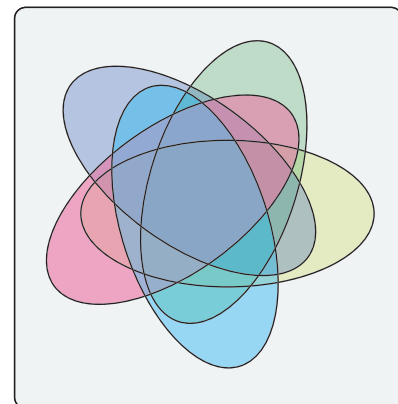
Figure 22 (see page 92) plots this argument, and we can see that the result is true.

Once I had the Edwards diagram programmed, I tried out a few variations. In Figure 23a I pulled the big blobs in close to the main circle using circular arcs. I also gave it a shot using Bezier curves, as in Figure 23b.

I also tried replacing the inner circle with a box. Figure 24 shows the results, where I again used arcs and Bezier curves in Figures 24a and 24b. In Figure 24c I used straight lines to join up the regions.

### Counting cells

As I mentioned before, a diagram representing  $n$



**20** A five-element diagram made out of four congruent ellipses.

object classes will contain  $2^n$  cells. I call the number of elements in a cell that cell's *density*, so a cell with three items has a density of 3.

It's interesting to look at the patterns formed by the cells' density for different diagrams. Figure 25 (page 93) shows several density diagrams.

### Karnaugh maps

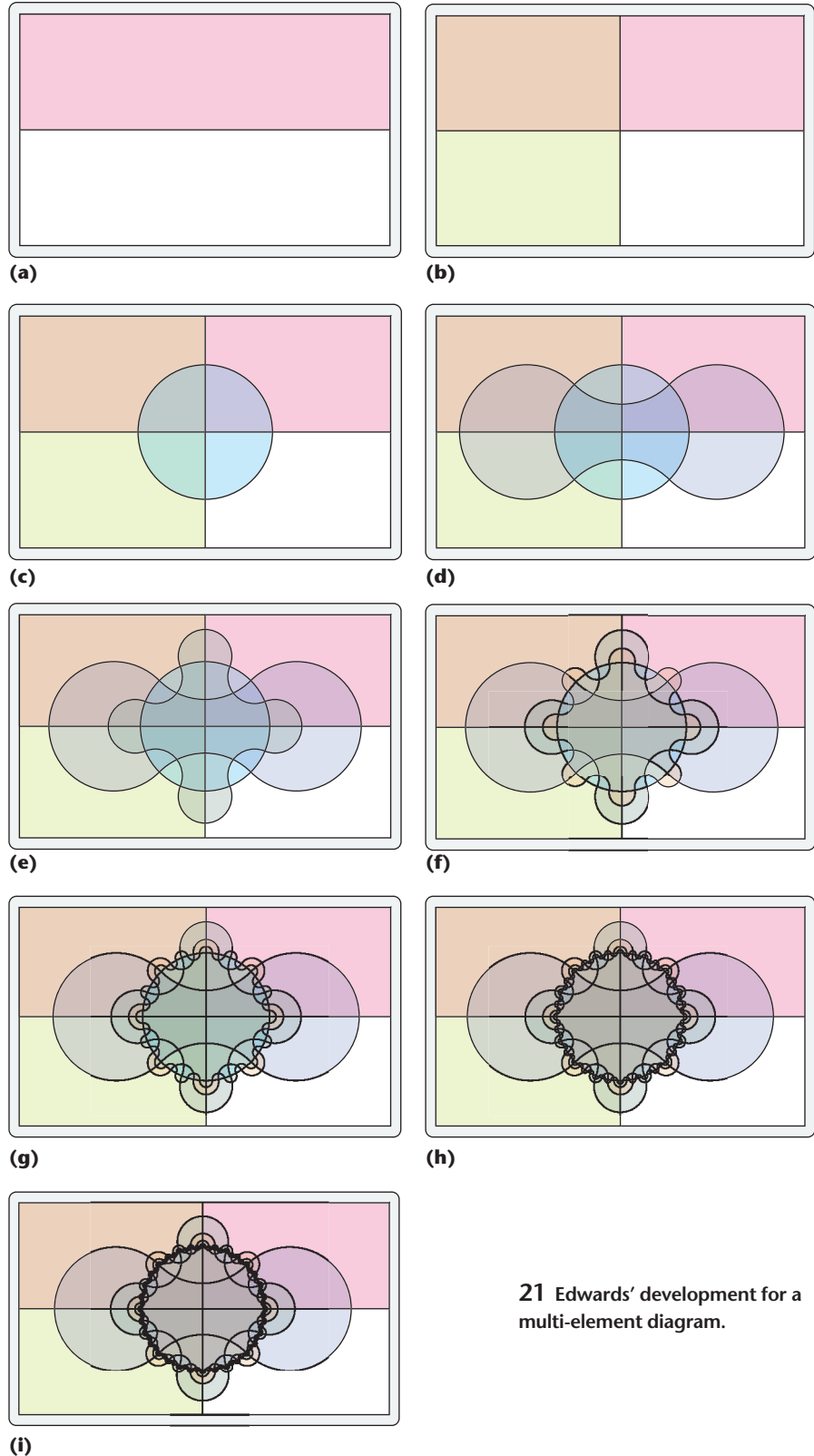
Related to Venn diagrams are a class of diagrams called *Karnaugh maps*. Electrical engineers typically use these to simplify Boolean circuits. Simplifying a circuit really just means finding a terse representation for the circuit's truth states, just as we try to verify a short statement of truth resulting from a logical argument.

Figure 26 shows the simplest Karnaugh map based on just two 1-bit variables,  $a$  and  $b$ . The trick with Karnaugh maps is to try to find the biggest rectangles of 0s and 1s that you can. If Figure 26 tells us the output of a circuit  $K_1(a, b)$ , then it's easy to tell that  $K_1(a, b) = a$ . We hardly needed any fancy tools for this example, so let's get a little more complex.

Karnaugh maps have only two dimensions, so to represent three variables we need to bundle them. Typically we treat the first two as a pair and the third as a single free variable, as in Figure 27. Notice that I laid out the possibilities for  $ab$  along the top line using a gray code, as we previously discussed. As we'll see, this is critical to the technique.

In Figure 27 it's easy again to spot the cluster of 1s, which I circled in red. If Figure 27 represents a circuit  $K_2(a, b, c)$  then  $K_2(a, b, c) = b$ . In other words, inputs  $a$  and  $c$  are irrelevant, which we can tell by looking at the diagram.

Now let's look at a circuit with four variables. As before, I'll pair them up and lay them out in gray code order, as in Figure 28 (page 94). I decided to simplify this circuit by identifying rectangles of 1s, and I found three such rectangles in Figure 28b. The purple and red rectangles are straightforward, but the four green regions don't look much like a single square. To see why we can consider them squares, take a look at Figure 28c, where I simply laid out multiple copies of the map side by side. The gray code layout makes this a valid approach because even across map boundaries we're only chang-

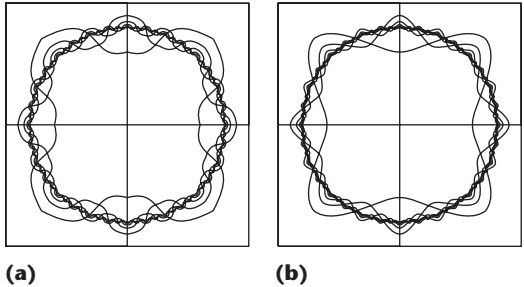
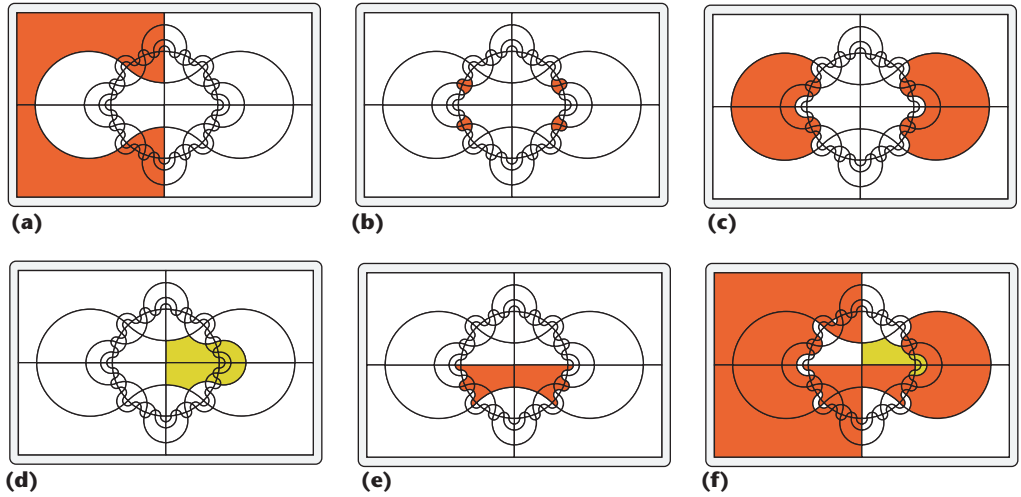


21 Edwards' development for a multi-element diagram.

ing one bit. In other words, you can pick any 4-by-4 block out of Figure 28c and you're still working with the original problem.

In this example, the red rectangle corresponds to  $c$ , the blue one to  $abc$ , and the green one to  $b'$ . So if Figure 28c is circuit  $K_3$  then  $K_3(a, b, c, d) = c + abc + b'$

**22** Argument plotted on Edwards' diagram. (a) No  $b'$  are  $d'$ , (b) no  $g$  are  $d f'$ , (c) no  $d$  are  $f'$ , (d) some  $b$  are  $ed$ , and (e) no  $d'$  are  $dg$ . (f) The five statements taken together prove the conclusion that no  $d'$  are  $b' c f g$ .



**23** Variation on Edwards' development for a multi-element diagram. Here I've kept the circle for the third element, but wrapped the other curves more closely around it. (a) Using circular arcs and (b) Bezier curves.

(remember that in this context, + means "or").  
Figure 29 (page 94) shows another pair of four-term examples. You can see that the larger the blocks, the simpler the resulting expression.

Figure 30 makes the connection apparent between Karnaugh maps and the logic diagrams we've used so far. Here I plotted the same data in four visual maps.

**Geometry**

If you like geometry, you might want to sit down and figure out how to make an Edwards diagram like the one I show in Figure 21.

I only had an image to work from and originally assumed that the curves were just hand-drawn lobes that looked nice. As I was drawing a version of the dia-

gram for this article, though, it seemed to me that all the curves were crossing the inner circle perpendicularly to the circle itself. To check this hypothesis, I wrote a program to draw the curves this way. The results look great, and it sure beats drawing the image by hand.

Figure 31 (page 95) shows the basic geometry. We start with a circle  $P$ , with center  $C_1$  and radius  $r_1$ , and two points on that circle, which I labeled  $A$  and  $B$ . Our goal is to find circle  $Q$ , with center  $C_2$  and radius  $r_2$ . Circle  $Q$  crosses  $P$  at points  $A$  and  $B$ , and is perpendicular to  $P$  at each crossing.

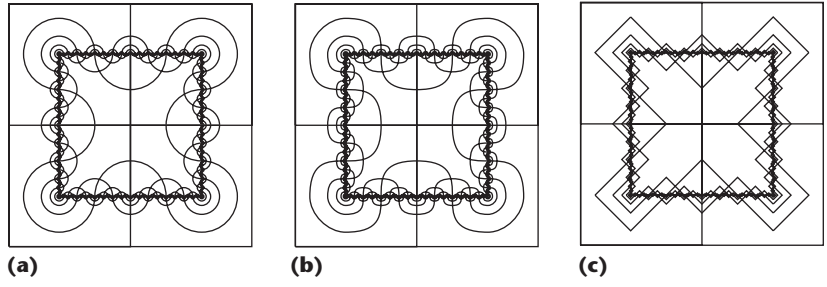
To find  $C_2$ , we can augment Figure 31a with some extra construction lines, as in Figure 31b. Symmetry tells us that  $C_2$  lies on a line that passes through  $C_1$  and point  $M$ , the midpoint of segment  $AB$ . From this diagram we can extract the triangle in Figure 31c. We want to find the angle  $\omega$ , which is easy because we know the lengths of all three triangle sides:

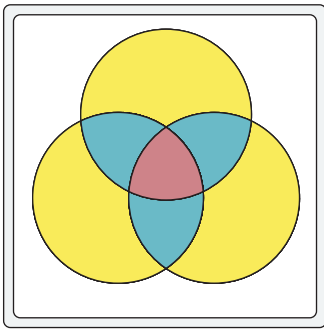
$$\omega = \tan^{-1} \left( \frac{|A - M|}{|M - C_1|} \right)$$

From Figure 31b we can also extract the triangle in Figure 31d. Note that  $\alpha = (\pi/2) - \omega$ . The Law of Sines is useful here; recall that it tells us that in any triangle, the ratio of the length of any leg to the sine of the angle opposite it is the same for all three legs. Using this, we can write

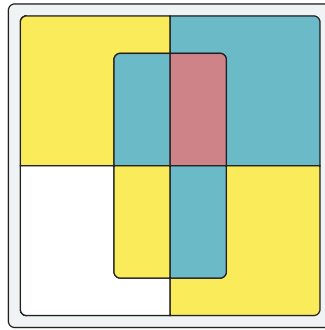
$$\frac{r_2}{\sin \omega} = \frac{r_1}{\sin \alpha} = \frac{d}{\sin(\pi/2)}$$

**24** My variants on Edwards' development in Figure 22. (a) Using arcs for the curves, (b) Bezier curves, and (c) straight lines.

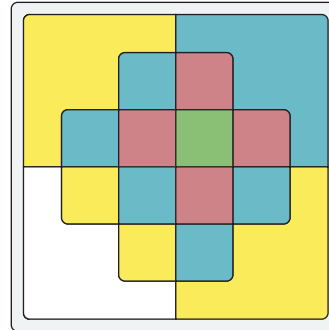




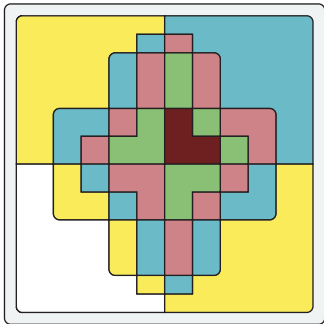
(a)



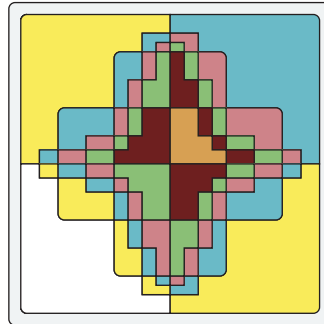
(b)



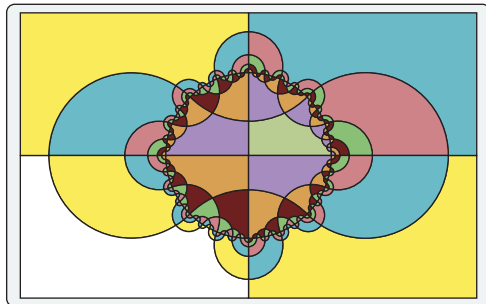
(c)



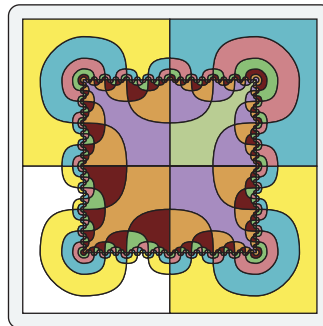
(d)



(e)



(f)



(g)

**25** Identifying the number of elements in each region. White represents zero elements, then yellow, blue, red, green, maroon, and orange represent one to six elements, respectively. The following are Venn diagrams: (a) three-element circular, (b) three-element rectangular, (c) four-element rectangular, (d) five-element rectangular, and (e) six-element rectangular. (f) The eight-element Edwards diagram. (g) One of my variants on the eight-element Edwards diagram.

		<i>a</i>	
		0	1
<i>b</i>	0	0	1
	1	0	1

(a)

		<i>a</i>	
		0	1
<i>b</i>	0	0	1
	1	0	1

(b)

		<i>ab</i>			
		00	01	11	10
<i>c</i>	0	0	1	1	0
	1	0	1	1	0

(a)

		<i>ab</i>			
		00	01	11	10
<i>c</i>	0	0	1	1	0
	1	0	1	1	0

(b)

**26** A two-term Karnaugh map  $K(a, b) = a$ . (a) Map and (b) a rectangle of 1s.

Since  $\sin(\pi/2) = 1$ , we have

$$d = \frac{r_1}{\sin \alpha} = \frac{r_1}{\sin\left(\left(\frac{\pi}{2}\right) - \omega\right)}$$

**27** A three-term Karnaugh map  $K(a, b, c) = b$ . (a) Map and (b) square of 1s.

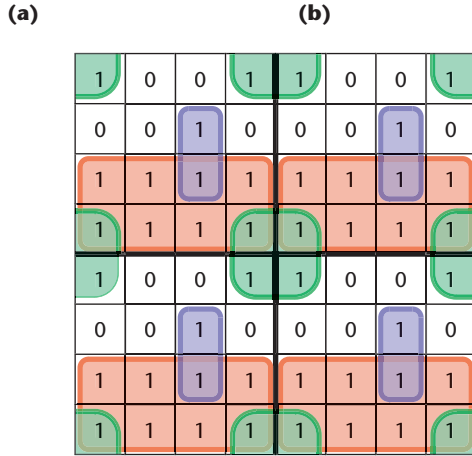
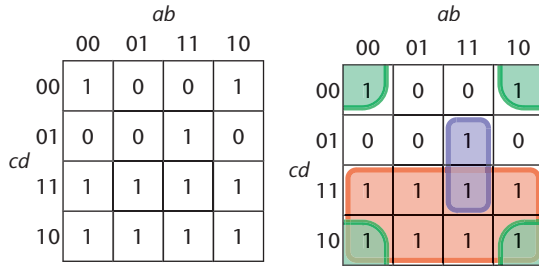
This gives us the distance  $d$  from  $C_1$  to  $C_2$ . Now we find the radius  $r_2$ , which also comes from the Law of Sines relation:

$$r_2 = d \sin \omega$$

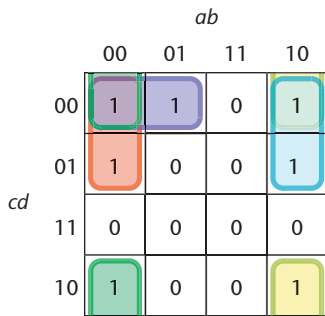
To actually find the center  $C_2$ , we just find the vector from  $C_1$  to  $M$ , normalize it to unit length, and then scale it up by the distance  $d$ :

**28** A four-term Karnaugh map  $K(a, b, c, d) = c + abc + b'$ .

(a) Map. (b) A 2-by-4 rectangle, a 1-by-2 rectangle, and a 2-by-2 square of 1s. (c) Seeing the green squares in Figure 28b by placing copies of the map side by side.



(c)



(a)

$$a'b'c' + a'c'd + a'b'd' + ab'c' + a'b'd'$$

(b)

$$(ab+bd+cd+bc)'$$

**29** Two more four-term Karnaugh maps. (a)  $K(a, b, c, d) = a' b' c' + a' c' d' + a' b' d' + a b' c' + a b' d'$  and (b)  $K(a, b, c, d) = (ab + bd + cd + bc)'$ .

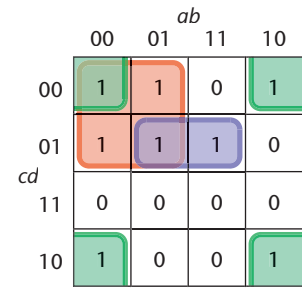


$$C_2 = C_1 + d \frac{M - C_1}{|M - C_1|}$$

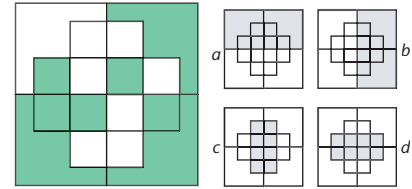
Now that we know  $C_2$ , we can draw the Edwards diagram. We just choose alternating arcs (the one inside the circle and the one outside) as we work our way around the inner circle, stepping points  $A$  and  $B$  in equal increments.

You can create as many generations of the Edwards diagram that you want by simply chopping the big circle into smaller pieces. I created all of the Edwards-style figures in this article with this algorithm.

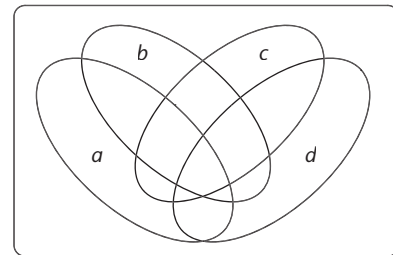
Debugging graphical programs like this can be as much fun as making them. While I was getting a generalized version of this to work for Figures 21 through



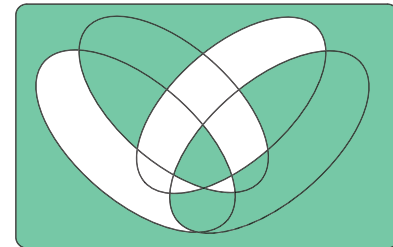
(a)



(b)



(c)



(d)

**30** The same four-element data in four different visual representations. (a) Karnaugh map, (b) Carroll's quadrilateral diagram, (c) Venn's four-ellipse diagram, and (d) Edwards' four-element diagram.



24, I generated a whole lot of interesting outtakes; Figure 32 shows some of my favorites.

### 3D

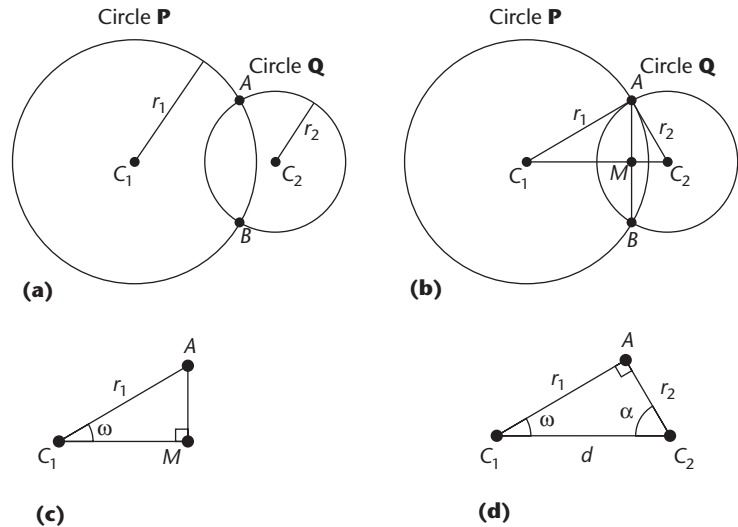
Now that we've talked about creating Venn diagrams in the plane, how about going into 3D? Are there any interesting developments to be found by drawing our diagrams on the surface of 3D objects such as spheres and donuts? How about creating bonafide 3D diagrams, say by replacing the circles of the classic three-Venn diagram with spheres?

I've toyed around with these ideas a bit, and hope to report on what I've learned in a future article. Until then, I invite you to think about it yourself and see if you come up with any new insights. ■

### Acknowledgments

Thanks to Steven Drucker for helpful advice, and my brother Bruce Glassner for a thoughtful reading of this article and excellent suggestions.

Readers may contact Andrew Glassner at [andrew@glassner.com](mailto:andrew@glassner.com).



**31** The geometry for Edwards' construction of Figure 21. (a) The main circle **P** and the two points **A** and **B** that we're given as well as the circle **Q** that we want to find. (b) Some labels. Point  $M = (A + B)/2$ . (c) Triangle extracted from (b). (d) Another triangle from (b).

### Further Reading

One of my primary references for this article was Frank Ruskey's excellent article, "A Survey of Venn Diagrams" (*The Electronic J. of Combinatorics*, February, 1997). Ruskey goes into much more detail on many of the topics I've covered here. It's available online at <http://www.combinatorics.org/Surveys/ds5/VennEJC.html>.

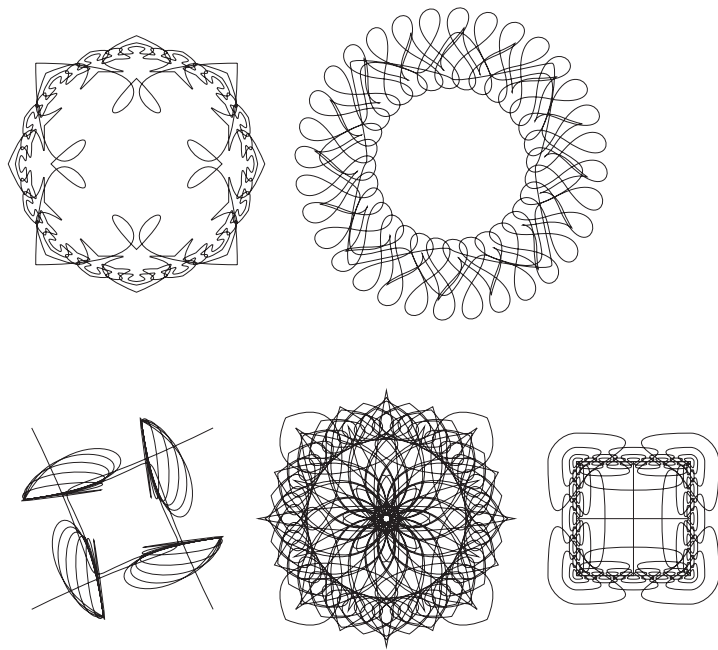
Branko Grünbaum's ellipses in Figure 20 were originally published in his article, "Venn diagrams and Independent Families of Sets," (*Mathematics Magazine*, vol. 48, Jan.–Feb. 1975, pp. 12-23).

The construction of Figure 21 due to Anthony Edwards appears in his "Venn Diagrams for Many Sets" (*New Scientist*, 7 Jan. 1989, pp. 51-56).

Karnaugh maps were first introduced in Maurice Karnaugh's classic paper, "The Map Method for Synthesis of Combinational Logic Circuits" (*Trans. Am. Inst. Electrical Engineers*, part 1, vol. 72, no. 9, 1953, pp. 593-599).

Lewis Carroll's *Symbolic Logic* was originally published in 1896. It has been reprinted in a single volume called *Symbolic Logic and Game of Logic* (Dover Publications, 1955). The second half of this book contains hundreds of syllogisms; the author intended to help readers develop fluency with the notions and the mechanics by making syllogism solving a two-player competitive game.

My quotes from Lewis Carroll's books *Alice's Adventures in Wonderland* (1865) and *Through the Looking-Glass* (1872) came from the public-domain editions of these classics at Project Gutenberg: <http://www.gutenberg.net>.



**32** Some outtakes. These pictures came up while I was debugging the program that drew Figures 21 through 24. It's always surprising to me how simple changes in the code can produce such different results.