Andrew Glassner's Notebook

Crop Art, Part 1

Andrew Glassner

What's the coolest display medium for computer graphics that you've ever seen? There's a medium out there that people don't often think about: a field of living, growing crops. The medium is high resolution and capable of displaying large images viewable from dozens of yards or meters. That's the good news. The bad news is that it has limited color fidelity. In fact, there are only two colors available, and you don't even get to pick them. Furthermore, making an image requires either hours of exhaustive effort with a small team, or the cooperation of friendly space aliens with spaceships and other technology.

Not any field of crops will do: Thanks to how easy they are to bend—and their willingness to stay bent—canola, wheat, barley, and oats are your best choices for efficient construction and crisp presentation. The general idea is to create a two-color design by flattening down some of the crop, as shown in Figure 1. The flattened parts will have a different color than the parts that are left standing. Using some form of technology (simple, complex, or extraterrestrial), you flatten the crops according to a pattern, and you've created your output.

These types of patterns are often called *crop circles* because the earliest examples were indeed simply circles, or small collections of circles. Today, crop art has moved far beyond simple circles, as Figure 1 shows. Sometimes crop designs are called *pictograms* or *formations*.

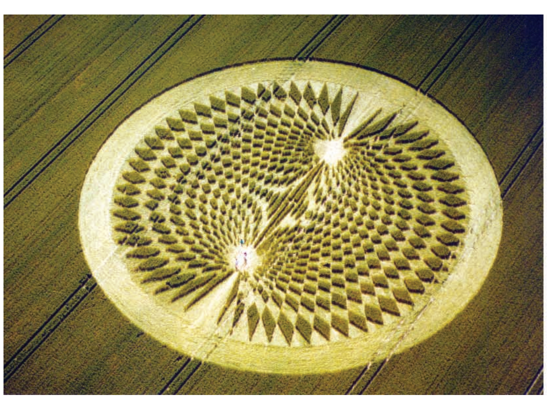
The subject of crop circles is fascinating in two ways. First, they pose interesting geometric challenges by the limited number of tools that you can typically use when creating such a design in the field, typically under the cover of darkness in a single summer evening. Second, the social phenomena surrounding these designs include a rich mixture of people who create them, who study them, and who vigorously debate a wide variety of conflicting theories regarding their creation and meaning.

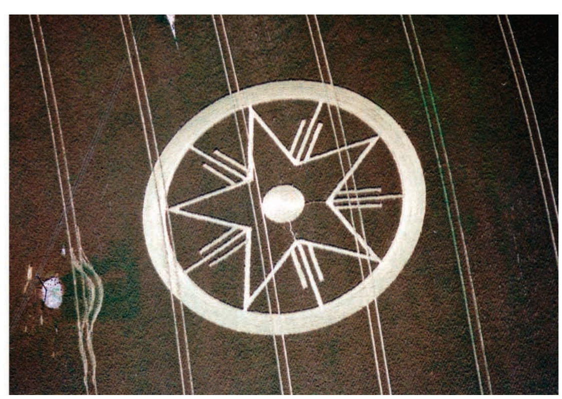
In this column, I'll look first at some of the interesting geometry behind crop circles and related formations. (To learn about the basic history of crop circles, see the sidebar "Cerealogy: A Balanced Breakfast.") Then, in the next issue, I'll talk about a language I've developed to make it easier to design and construct crop circles. Finally, I'll discuss some of the practical aspects of making crop circles, illustrated with my experience of actually making a formation.

Basic construction

Let's suppose that we would like to make a crop-circle design. If we're to go about it in the traditional way, we have a few constraints to obey.

The most important limitations are that we need to make our construction under the cover of darkness, over the course of a single evening. To avoid detection, we can't use flashlights, lasers, or any other large light-emitting devices. We want to leave behind as little evidence of our presence as possible, so that means not only no trash, but no accidentally crushed stalks and no holes





1 **Two actual crop circles. Note that there are only two colors available in the design. (Images copyrighted by Colin Andrews, http://www.CropCircleInfo.com, and used here with permission.)**

Cerealogy: A Balanced Breakfast

There's no way to know exactly when or where the crop-circle phenomenon started. It certainly goes back to at least 15 August 1980, when the British paper the *Wiltshire Times* published an account of three circular shapes in England's oat fields. Over the next few years, circles began to appear in other English fields, and local papers would run stories along with a photograph or two. Nobody really knew what to make of them, although there was widespread speculation that they were somehow associated with, or even created by, space aliens. Groups of researchers started to form, dedicated to studying each new formation as it was discovered.

On 9 September 1991, the British tabloid *Today* published an article titled "The Men Who Conned the World." In the article, two landscape artists named Doug Bower and Dave Chorley announced that they had been behind many of the crop circles that had been observed in the previous decade. The next day the two men, respectively 67 and 62 years old, demonstrated their technique at a live press conference in Chilgrove in Sussex. Using a couple of boards with ropes, they created a pair of dumbbell-shaped formations for a group of reporters, who filmed the entire demonstration.

Anyone who thought that this would put an end to the theories that the circles were of extraterrestrial origin would have been surprised. Although some accepted this demonstration, many of the crop-circle groups rejected both the claims and the demonstration as fakes. It's an interesting point of view: Two men who came forward as the perpetrators of a hoax were declaimed themselves as hoaxsters of a higher order, since their claims to have created the hoax were seen as bogus.

Cerealogists

This set the stage for what is now standard terminology used by those who study crop circles, called *cerealogists*. In cerealogist terminology, a genuine crop circle is one that was created by space aliens, using any of a wide variety of possible technologies, from radiation or controlled whirlwind fields, to ball lightning, Q rays, and undetectable manipulations of matter. On the other hand, formations created by people are termed hoaxes, frauds, and fakes. The people who make such formations call themselves *circlemakers*, but cerealogists call them hoaxsters and frauds.

Cerealogists have developed a number of techniques to determine for themselves whether a given crop formation was made by humans, or left behind by aliens. These tests include rubbing stalks of grain together and noticing which cardinal direction they seem to move toward, using dowsing rods on the area, and measuring for unusual levels of phenomena ranging from bacterial activity to radioactivity. Cerealogists also investigate the nature of the bent and broken crops themselves, reasoning that certain patterns of flattening, and specific types of damage to the stalks, are impossible for humans to create. Formations that demonstrate those patterns must therefore be genuine (that is, of extraterrestrial origin).

Circlemakers

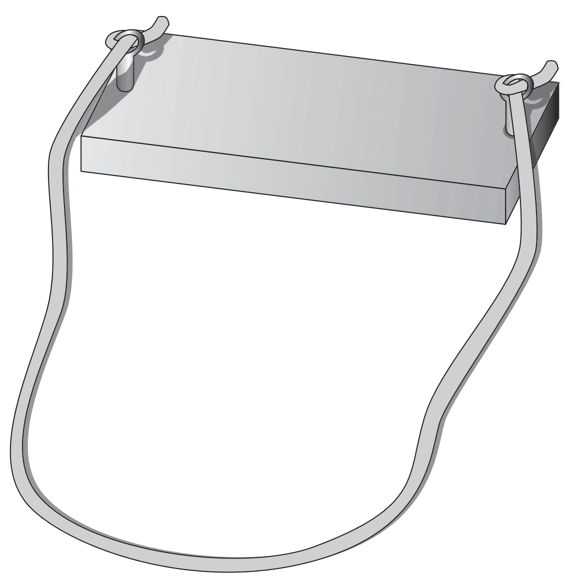
Self-described circlemakers are motivated by a wide range of impulses. One of the most common of these is the desire to create large and impermanent art. They share an aesthetic with artists who build complicated sand castles at the beach: The fun is in the process, and in the knowledge that the art itself is fleeting and will soon disappear.

The sociology of crop circle creators, investigators, and adherents of each theory is a fascinating symbiosis: The investigators wouldn't exist without the formations to look into, the theorists couldn't argue without measurements to refer to, and it's likely that many of the artists wouldn't bother to make the works if there wasn't such an attentive and appreciative audience out there to receive it. The fact that most circlemakers work anonymously is probably because of Doug's and Dave's legacy, which set the tone for overnight, stealth construction.

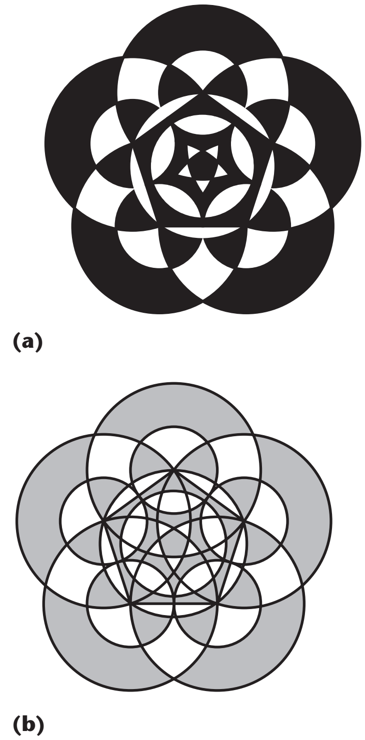
left over from posts and stakes pushed into the ground. Traditionally, we go out with nothing but some surveyor's tape, string, a plan, and a *stomper*. A stomper is a tool for flattening grain.

Like many elegant tools, the stomper is simple. It was originally shown by Bower and Chorley in their famous 1991 demonstration. A typical stomper is shown in Figure 2. It's just a single board, usually about three feet wide, with a six-foot length of rope knotted or tied near the ends of the board. To flatten a chunk of crop, you hold the rope handle in your hands, put your leading foot on the stomper, and press down. Then you bring your trailing foot up to just behind the stomper, and take another step with your leading foot, flattening another swatch of grain. And that's all there is to it. Just stomp, step, stomp, step, stomp, until your design is complete.

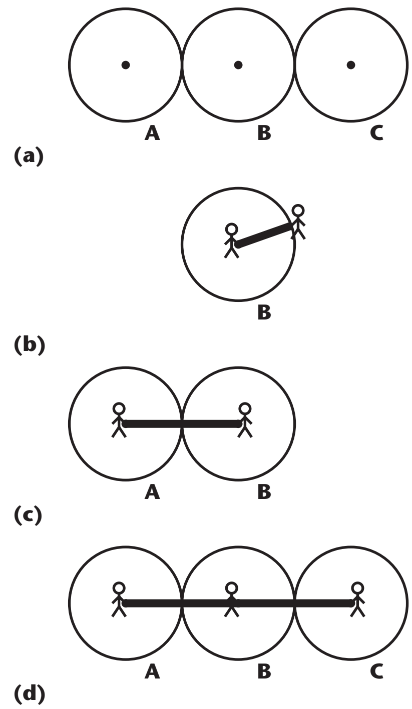
So how do you use a stomper to make a design? Generally crop formations are made of straight lines, circles, and arcs. A pattern usually begins when you establish one or more construction marks to guide your later stomping. For example, to make a circle you'd have



2 **A stomper for flattening crops.**



3 (a) Original crop formation. (b) Schematic behind the formation.



4 (a) We'd like to make these three equal circles. Their centers lie on a straight line. (b) We can make the center circle anywhere. (c) To find the center of circle **A**, simply pull a string of radius 2*r* from the center of **B** to any point. (d) To find the center of circle **C**, hold one end of a string of length 4*r* at the center of **A**. Have a friend stand at the center of **B** to make sure it passes through that point. The other end is the center of **C**.

a friend stand at the center of the circle, holding one end of a piece of string. Facing your friend, you'd pull the string taut and then step sideways, keeping the string pulled tight. You'll push down the stalks under your feet as you walk, eventually creating a thin ring. You can then use the stomper to flatten down the ring's interior.

Depending on how you choose to flatten the crops inside the circle, the lay of the flattened grain can form concentric circles, a spiral, a woven thatch, or any other pattern you like. Most designs are made out of simple geometric elements. We use a few key points to guide lines and arcs, and then we stomp down regions defined by those lines and arcs.

We can describe a final formation like Figure 3a with the schematic of Figure 3b. Once all of the construction lines have been laid down, you just stomp down the interiors of the regions that you want filled.

There's an important caveat here, which affects this technique in practice: there's no eraser. You can't remove your construction lines after you're done. This means it's important to create designs where all of the construction lines are eventually incorporated into the design itself. Any stray or leftover construction lines will show up in the field. Of course, if you like them and feel that they're part of the pattern, that's fine, but one traditional mark of a quality construction is that it has no visible artifacts of its construction.

Suppose you want to make a trio of three equal circles in a row, as in Figure 4a. How might you do this? Here's one way. You could create one of the circles just about anywhere, as in Figure 4b. This will be the center circle, marked **B** in the figure. Let's say it has radius *r*. To make circle **A**, one person stands in the center of **B** holding the surveyor's tape, and another person walks away, keeping the tape taut, until he reaches a distance 2*r*. That's the center of circle **A**, as in Figure 4c. To find the center of circle **C**, one person holds one end of the tape standing in the middle of **A**, another person holds it loosely at the center of **B**, and a third person pulls it taut to a distance of 4*r*. That's the center of **C**, as Figure 4d shows.

When the designs get more complicated, developing a precise construction plan becomes critical to successfully making the design without error, particularly given the usual constraints of darkness, not enough time, and not enough people. Finding key points for the centers and ends of arcs and lines has to be efficient and accurate.

The limitations of working in the field create some interesting limits on our designs. In the next section, I'll focus strictly on the pencil-and-paper stage of crop-circle design, when we're still indoors, dry, and warm. What makes the process interesting is that we need to always keep in mind our limited capabilities when we're actually out in the field.

Hawkins' five theorems

Let's look at some of the design problems related to the geometry of traditional crop formations.

Since photographs of crop circles first started appearing in newspapers and magazines in 1980, people have been assembling archives in their homes, books, or most recently on the Web. Several of the sites in the "Further Reading" sidebar offer huge galleries of gorgeous photographs of crop-circle formations.

One person who was fascinated by the geometric regularity of many crop circles was Gerald Hawkins. Hawkins was a professor of physics and astronomy at Boston University in Massachusetts, publishing a steady stream of technical papers in research journals. In 1961, he traveled with a group of students to Salisbury Plain in England, the site of Stonehenge. Together, they measured every stone, rock, pit, and formation at the site. Upon returning home, Hawkins used an IBM 704 computer to help him analyze the data and look for patterns.

He concluded that Stonehenge was an observatory, designed to predict eclipses, solstices, and other celestial phenomena. He published his argument in a 1963 paper in *Nature*. The paper created a wave of public interest and made him a famous popular figure. But the paper created an almost immediate backlash of objections from archaeologists and astronomers, who rejected both his methodology and his conclusions.

In the late 1980s, Hawkins found himself fascinated by crop-circle formations. Working from published photos, he measured everything that seemed measurable and then hunted through the data, looking for patterns.

After much measuring and searching, he started to find some relationships. Abstracting the measurement of some formations into their underlying geometry, he found nice whole-number ratios between various measures. This excited him because these ratios (such as 2:1 and 4:3) are also the ratios between notes in a well-tempered major musical scale. Feeling that he was onto something, Hawkins organized his observations as five theorems, which I'll present in a bit.

Hawkins believed that at least some of the people who were making new designs had knowledge of his theorems, and were using them actively to design their formations. The implication was that they must have independently discovered his theorems, because otherwise they would have been unable to produce those particular formations. As a prime example, he pointed to a formation in Guildford, England (see Figure 5). The strange thing is that Hawkins claimed that these formations actually *proved* that the designers knew about his theorems, and were using them actively.

Of course, the pattern in Figure 5 proves nothing. Anyone can draw this simply by drawing an equilateral triangle, and then setting the size of a compass by eye to draw inscribed and circumscribed circles. If you chose to actually compute the radii of those circles (as we'll do later), you could easily do it with standard first-year trigonometry. The design of Figure 5 could indeed have been made with Hawkins' First Theorem, as he asserted, but it could just as easily have been designed with a computer-aided drafting program, a ruler and a compass, or a beer mug, a coin, and the side of a square coaster.

In a 1992 *Science News* article, Hawkins described his first four theorems, and alluded to a fifth. The article said that Hawkins was "inviting anyone interested to come up with the theorem itself before trying to prove it." We'll discuss the fifth theorem later, but there's nowhere near enough information in the article to have any idea what that fifth theorem could be. Asking people to come up with the theorem is like an author challenging his audience to discover the plot of his next

Further Reading

There are lots of books available on crop circles if you'd like to investigate the subject further. I drew some of my historical overview from *Crop Circles*, by W. Anderhub and H.P. Roth (Lark Books, 2000). This book also contains a wealth of nice images.

For more practical matters, I used a small pamphlet, *A Beginner's Guide to Crop Circle Making*, by R. Irving, J. Lundberg, and R. Dickinson (Circlemaker's Press, 2004).

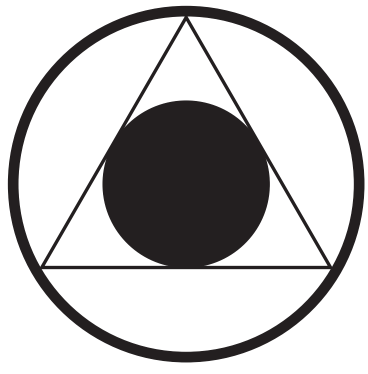
Gerald Hawkins' connections to crop circles probably started in the 1992 feature article "Off the Beat: Euclid's Crop Circles" by I. Peterson, *Science News*, vol. 141, Feb. 1992, pp. 76-77 (available online at http://www.gaiaguys.net/Science_News_2.92.htm). A slightly revised version appeared as "Theorems in Wheat Fields" by I. Peterson, *Science News,* vol. 163, no. 26, 28 June 2003 (available at http://www.sciencenews.org/articles/20030628/mathtrek.asp).

Follow-up articles were "Geometry in English Wheat Fields" (unattributed, possibly by H.B. Tunis), *The Mathematics Teacher,* vol. 88, no. 9, Dec. 1995, p. 802, and "From Euclid to Ptolemy in English Crop Circles" by G.S. Hawkins, *Bulletin of the American Astronomical Society*, vol. 23, no. 5, p. 1263, 1997. The unattributed advertisement I mentioned in the text is frequently referenced in crop circle literature as a formal publication, but it's clearly labeled as an advertisement. It appears with the title "New Geometry in English Crop Circles" in *The Mathematics Teacher*, vol. 91, no. 5, May 1998, p. 441. Another follow-up article was "Crop Circles: Theorems in Wheat Fields" by I. Peterson, *Science News*, 12 Oct. 1996 (available online at http://www.gaiaguys.net/ffgeom.htm).

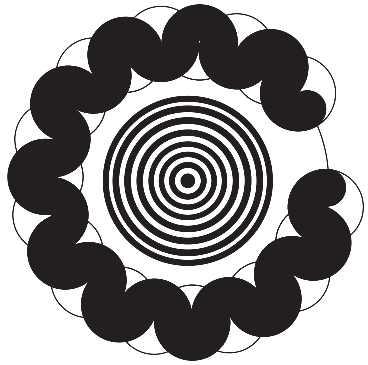
Dozens of Web sites are devoted to crop circles. Some good sites to get a running start into the subject include (in alphabetical order)

- http://www.circlemakers.org,
- http://www.bltresearch.com,
- http://www.cropcircleconnector.com/anasazi/connect.html,
- http://www.invisiblecircle.org,
- http://www.fgk.org,
- http://www.kornkreise.ch,
- http://www.lovely.clara.net, and
- http://www.swirlednews.com.

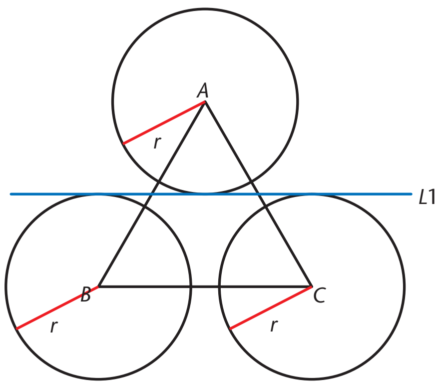
Zef Damen has published many geometric reconstructions of actual formations on his Web site. I learned a lot by studying his presentations. Visit http://home.wanadoo.nl/zefdamen/en/Crop_circles_en.htm for links to his reconstructions.



5 Schematic of a formation in Guildford, England, which Hawkins asserted proved knowledge of his first theorem.



6 A formation which Hawkins asserted proved knowledge of his fifth theorem (after an image in "Crop Circles: Theorems in Wheat Fields").



7 Setup for Hawkins' first theorem. Three circles of radius *r* are centered on the vertices of an equilateral triangle. Their radius *r* is chosen so that all three circles share a common horizontal tangent.

novel, and then summarize it; there's just not enough information available to get started on the problem. But because he had phrased this as a challenge, it got people's attention.

Thanks to this air of mystery, the fifth theorem started to take on a life of its own among cerealogists, who began to view it as a piece of secret information communicated to Hawkins by extraterrestrials. Hawkins couldn't publish the theorem, the story went, because it was either too complex for people to understand, or the information was too dangerous for humans to know at our current stage of evolution. Hawkins was keeping it a secret for the good of humankind, holding it back until our species could show we were ready for it.

The fires of this story were fanned when *The Mathematics Teacher* published a one-page article on Hawkins' theorems. The article was only six paragraphs long, with an equal number of references, but it offered a summary of his five theorems. Theorems one through four are pretty straightforward. But as with the *Science News* account, the article only said that theorem five was a generalization of theorems 2, 3, and 4 without any more details. It concluded by inviting readers to create and prove Theorem 5 and send two copies of the proof to the magazine. Again, there was little chance of that happening given the information in the article. But again, it appeared that a gauntlet had been laid down, and the best mathematical minds on the planet had been stumped.

The longer these two challenges remained unanswered, the larger the myth of Theorem 5 became. All sorts of improbable ideas began to get associated with the power of this unknown, but presumably vital, geometric relationship. Perhaps, some speculated, it held the clues for world peace, nuclear fusion, or even time travel.

Fifth-theorem fever got another boost in 1996, when *Science News* ran a follow-up article on Hawkins' Theorems. This article contained a couple of figures that were described as relating to the now-legendary Theorem 5. But the mythology got cranked up another notch, because the article presented a photograph of a formation found in the field which led Hawkins to assert the circlemakers were demonstrating knowledge of his fifth theorem. The implication was that someone had answered his challenge, and presented their proof to him by creating the formation. Figure 6 shows this design. Once again, this formation doesn't prove anything; it's simply a nice pattern that could be produced by anyone handy with a compass and ruler.

In 1998, a company identified as *Boston University Research* took out a quarter-page advertisement in *The Mathematics Teacher*. The advertisement, which didn't name an author (but which was presumably written by Hawkins) reiterated the familiar challenge. The article provided a schematic drawing labeled "V," but again no information actually tells what this fabled fifth theorem might be.

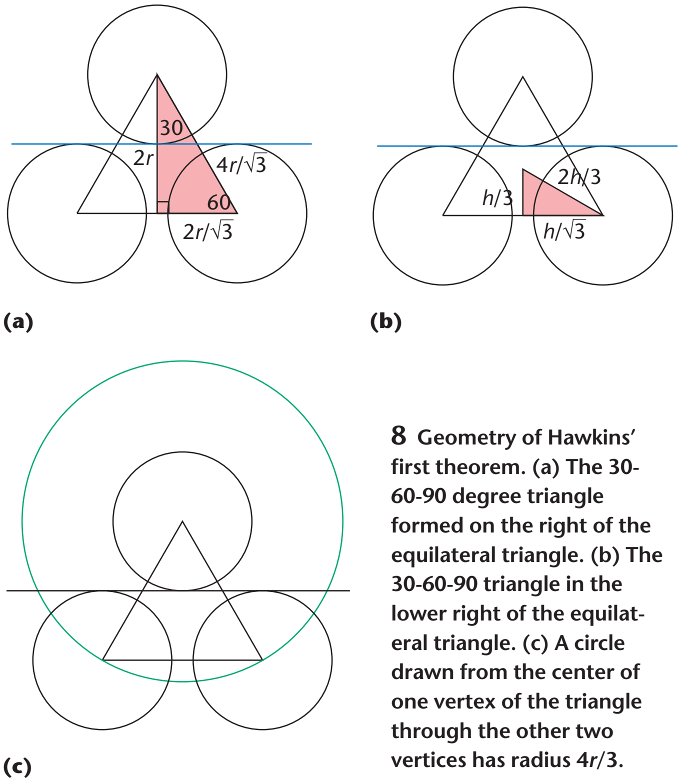
Hawkins died at the age of 75 on 26 May 2003. As far as I could tell after exhaustively searching online, with extensive help from a University of Washington research librarian, Hawkins never published his fifth theorem. However, we found an old copy of an unattributed homework handout on the Web (which has since disappeared) that mentioned Hawkins' fifth theorem in passing, with enough context to see what it was all about. Once you know what it is, it seems that Hawkins was playing a game with his audience, giving them just enough information to let their speculation run unchecked, while withholding enough information to make sure that nobody could answer his challenge.

Now that we know the history behind Hawkins' five theorems, let's actually look at them. Each time these theorems appeared in print he described them in a different way, sometimes with typographical or other errors. He also illustrated them with different figures, some of which are utterly at odds with each other. For such simple geometrical statements, this confusing wealth of contradictory and incomplete detail is maddening.

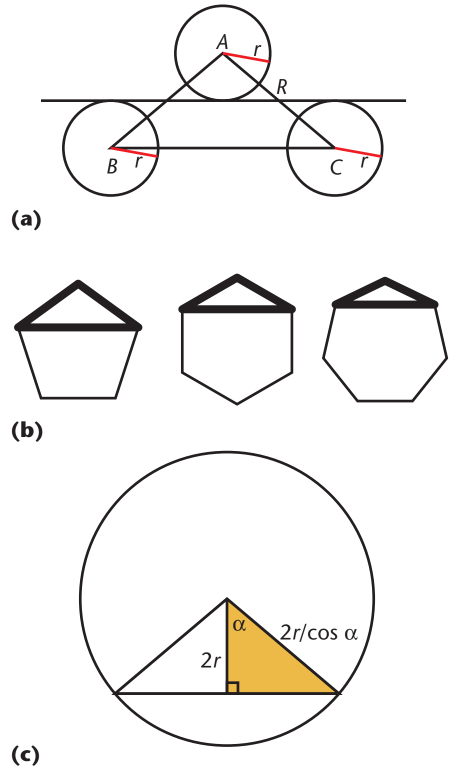
For the sake of clarity I'll present them here in a single consistent form. I also won't try to prove them as rigorous theorems, since I think they fit much more in the class of observations. I'll provide proofs that are only as rigorous as necessary to be convincing.

Hawkins' Theorem 1

Theorem 1. Place three circles of equal radii at the corners of an isosceles triangle, and choose their radius *r* so that they share a common tangent. Draw a circle **C** centered on one vertex of the triangle and passing through the other two vertices:



8 Geometry of Hawkins' first theorem. (a) The 30-60-90 degree triangle formed on the right of the equilateral triangle. (b) The 30-60-90 triangle in the lower right of the equilateral triangle. (c) A circle drawn from the center of one vertex of the triangle through the other two vertices has radius 4*r*/3.



9 Generalizing Hawkins' first theorem for a regular *n*-gon. (a) Three points of the *n*-gon with three equal circles that share a common horizontal tangent. (b) That triangle for three different *n*-gons. (c) A circle drawn from one vertex through the other two has radius 2*r*/cos α.

a. The radius of the circumcircle of the isosceles triangle is $4r/3$.
b. The radius of circle **C** is $4r/\sqrt{3}$.

Let's start with the drawing in Figure 7. Each of the three circles has radius r, they sit on the corners of an equilateral triangle, and they share a common horizontal tangent. The height of the triangle is thus $2r$.

To prove part a, in Figure 8a I've extracted the right side of the equilateral triangle, which is a 30-60-90 degree triangle with a long leg of length $h = 2r$; thus the short leg of the triangle is $h/\sqrt{3}$, and the hypotenuse is $2h/\sqrt{3}$. In Figure 8b I've drawn another 30-60-90 triangle, finding that the distance from the triangle's center to one vertex is $2h/3 = 4r/3$. That's all there is to part a.

To prove part b, in Figure 8c I've drawn the required circle. We know that its radius is the length of one side of the triangle, which we know from the last step is $2h/\sqrt{3} = 4r/\sqrt{3}$, as claimed. That's it for part b.

We can actually generalize the first theorem quite nicely for any regular n-gon, not just triangles.

In Figure 9a I've drawn the situation for any regular n-sided polygon. We simply put the three circles on the vertices of the triangle as before, so that they share a common tangent.

Figure 9b shows the measures for the relevant triangles. The length of one side of the n-gon is $2r/\cos\alpha$. We know that the interior angle of a regular n-gon is $(\pi(n-2))/n$, so because α is half of that, $\alpha = \pi(n-2)/(2n)$.

The radius of the circle is then $2r/\cos\alpha$. Plugging in $n = 3$ gives us $4r/3$, as expected. We can easily find the radius of any other polygon with n sides. For a square ($n = 4$), the radius is $2r/\sqrt{2}$, and for a hexagon ($n = 6$) the radius is 4r.

We can also compare the area of this circle to the area of the circle circumscribing the polygon. The ratio of these areas is

$$\frac{\pi(2r/\cos\alpha)^2}{\pi r^2} = \frac{4}{\cos^2\alpha}$$

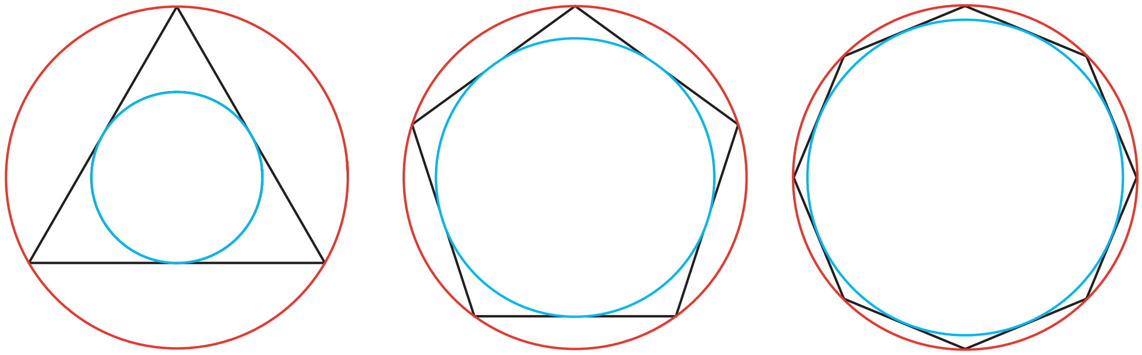
Plugging in $n = 3$ gives us $4r/\sqrt{3}$. For a square ($n = 4$) the ratio is 8, and for a hexagon ($n = 6$) the ratio is 16.

Hawkins' Theorem 2

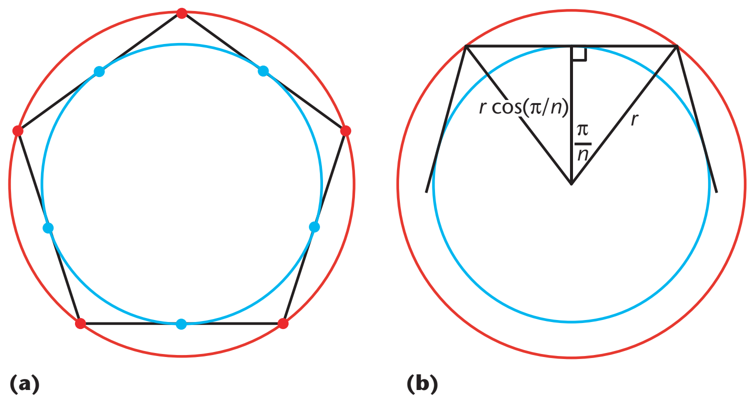
Before we get to Theorem 2, let's find a useful relationship. Take any regular polygon of n sides, and draw the smallest circle that encloses it (the circumcircle) and the biggest circle that fits inside (the incircle), as Figure 10 shows (next page). What's the ratio of the areas of these two circles?

Figure 11a shows the general idea. The circumscribed circle is centered at the polygon's center and passes through all of its vertices. The inscribed circle is also centered at the polygon's center, but it passes through the edges' midpoints.

Figure 11b shows the triangles formed by one side of the n-gon and the center. Since the n-gon spans 2π radians and it has n sides, each side spans $2\pi/n$ radians. Half of that, as shown in the figure, is π/n. This gives us the



10 **Circumscribed (red) and inscribed (blue) circles in a triangle, pentagon, and octagon.**



11 **(a) In a regular *n*-gon, the circumscribed circle passes through the vertices, while the incircle passes through the midpoints of the edges. (b) The distance from the center of the *n*-gon to the center of an edge is $r\cos(\pi/n)$.**

triangle as shown, showing that the distance to the center of the midpoint is $r\cos(\pi/n)$.

So the radius of the circumcircle is just r, and the radius of the incircle is $r\cos(\pi/n)$. The ratio of their areas is thus

$$\frac{\pi r^2}{\pi[r\cos(\pi/n)]^2}=\frac{\pi r^2}{\pi r^2[\cos(\pi/n)]^2}=\frac{1}{\cos^2(\pi/n)}$$

Okay, we're now set to dig into Theorem 2.

Theorem 2. The ratio of the area of the circumscribed and inscribed circles of an equilateral triangle is 4.

This is easy enough, just plug $n = 3$ into our formula:

$$\frac{1}{\cos^2(\pi/3)}=\frac{1}{(1/2)^2}=4$$

Hawkins' Theorems 3 and 4

The next two theorems are simple:

Theorem 3. The ratio of the area of the circumscribed and inscribed circles of a square is 2.

Plug $n = 4$ into our formula:

$$\frac{1}{\cos^2(\pi/4)}=\frac{1}{(\sqrt{2}/2)^2}=2$$

Theorem 4. The ratio of the area of the circumscribed and inscribed circles of a regular hexagon is 4/3.

You know the drill:

$$\frac{1}{\cos^2(\pi/6)}=\frac{1}{(\sqrt{3}/2)^2}=4/3$$

Hawkins' Theorem 5

And now (drumroll, please), the legendary fifth theorem!

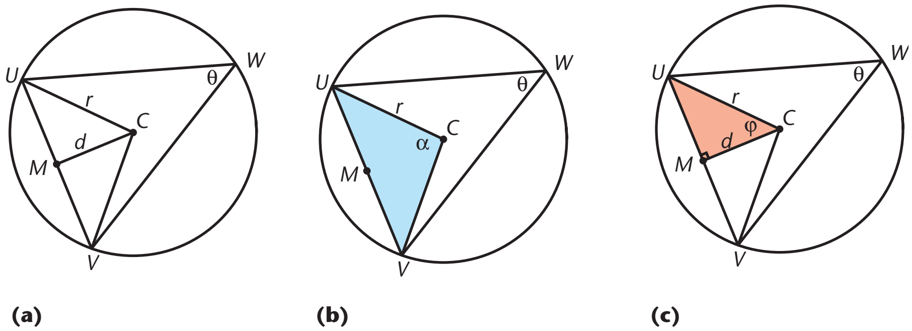
Theorem 5. Given any triangle inscribed in a circle of radius r, the perpendicular distance from the circle's center to any side of the triangle is $r|\cos\theta|$, where θ is the angle opposite the side.

You can see the geometry in Figure 12a. We have a triangle *UVW* inscribed in a circle with center *C* and radius r (the point *W* is chosen so that it does not lie on the minor arc connecting *U* and *V*). I've placed point *M* at the intersection of *UV* and the line that is perpendicular to *UV* that passes through *C* (note that *M* is not defined as the midpoint of edge *UV*). As instructed, I've labeled the angle opposite side *UV* at point *W* with θ, and I've marked the distance $d = |CM|$. The theorem says $d = r\cos\theta$. Let's prove it.

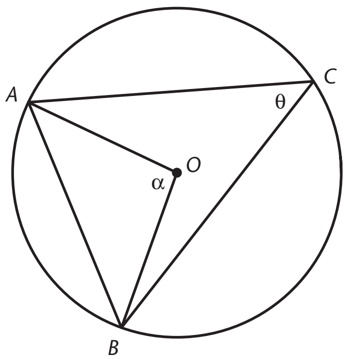
This is really easy if we remember the Inscribed Angle Theorem from basic geometry. It says that in a circle, any inscribed angle is half the corresponding central angle. Figure 13 shows this for a circle with center *O*, and three points *A*, *B*, and *C*, where *C* is not on the shortest (or minor) arc between *A* and *B*. The central angle formed by the arc *AB* is angle $\alpha = AOB$, while the inscribed angle is $\theta = ACB$. No matter how we choose the points, angle *AOB* will always be double that of *ACB*.

With this in our pocket, let's add a few more pieces of notation to Figure 12, giving us Figure 12b. I'll mark the angle *UCM* as ϕ. Since $d = r\cos\phi$, to prove the theorem we need to find ϕ.

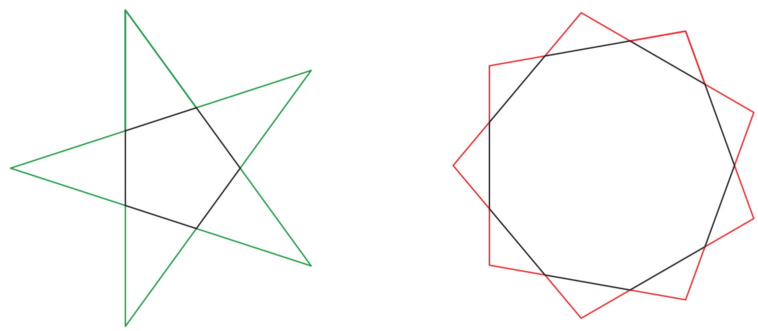
Now that we've got everything labeled, we can prove this theorem just by looking at Figure 12. The Inscribed Angle Theorem tells us that angle $\alpha = UCV$ is 2θ. From symmetry, we see $\phi = \alpha/2 = \theta$, so $d = r\cos\theta$, as claimed.



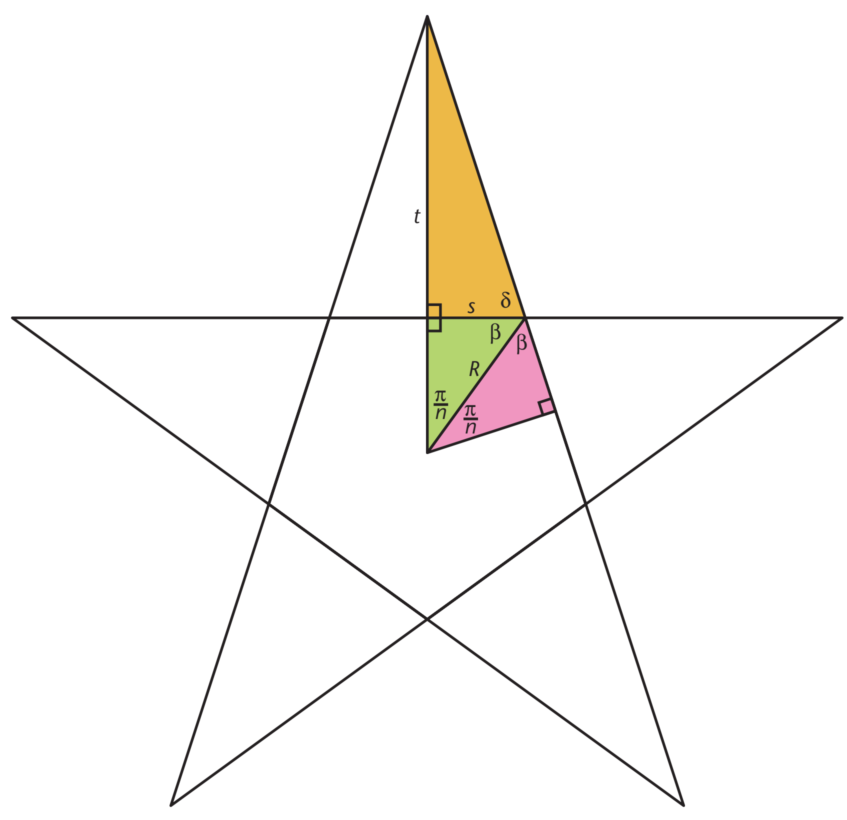
12 (a) The setup for Hawkins' fifth theorem. The points U and V sit on a circle centered at C with radius r. Point W is also on the circle, but not on the minor arc between U and V. The angle at W is θ. Point M is at the intersection of UV and the line perpendicular to it that passes through C. The distance $d = |CM|$. (b) The central angle $\alpha = UCV$. (c) The angle α is twice the angle $\phi = UCM$.



13 Inscribed Angle Theorem. Angle $\theta = ACB$ is half of $\alpha = AOB$.



14 Stellation of a pentagon and a nonagon.



15 Geometry for stellating a regular n-gon.

You'll notice that the theorem uses an absolute-value sign around cos θ. You can see the reason for that if you draw the triangle so that $\theta > \pi/2$; the absolute-value sign is just a way to save us from having to write the theorem in two cases.

Other useful geometry

The interesting Hawkins theorems are the first and the last ones. They can actually help us design interesting structures to flatten with our stompers.

Following in that spirit, I'd like to offer a few more useful design relationships of my own.

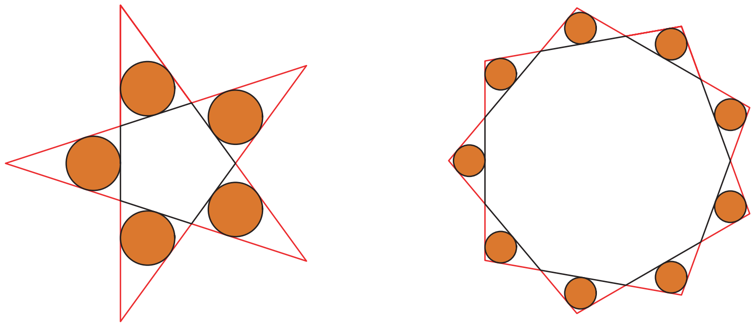
Stellate

Suppose you have a regular n-gon and you'd like to put points on each side, as in Figure 14. The points are found by extending neighboring pairs of sides until they meet.

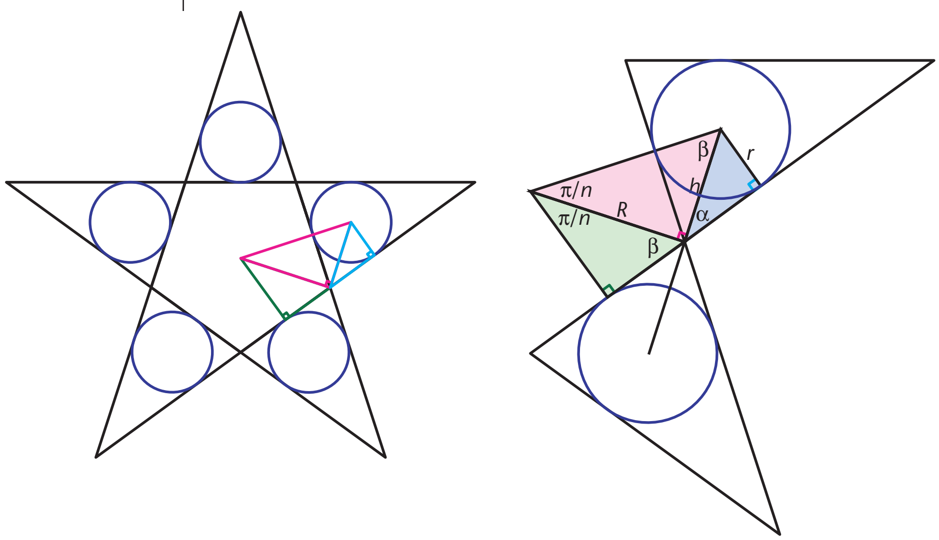
To find the point at the tip of each star, look at Figure 15. Here I've drawn a regular n-gon with $n = 5$ and radius R. The sides of the n-gon have a length of $2s$. From the green triangle, we can simply read off $s = R \sin(\pi/n)$. The third angle in the green triangle is β, which is what's left from the π radians in every triangle after we remove the right and central angles:

$$\begin{aligned}\beta &= \pi - (\pi/2) - (\pi/n) \\ &= \pi((n-2)/(2n))\end{aligned}$$

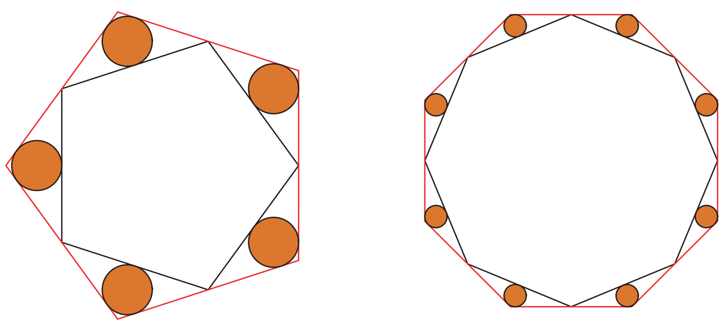
Moving to the orange triangle, we can see that β +



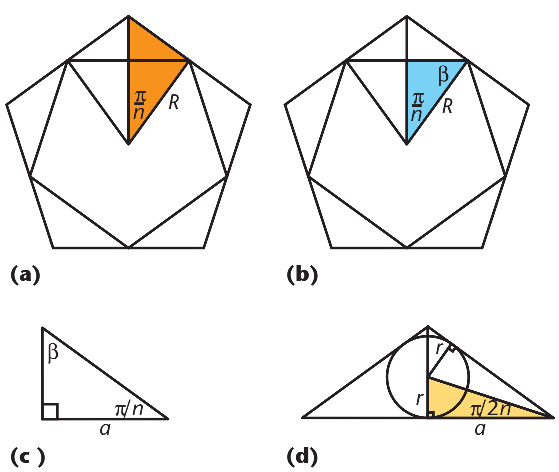
16 Star circles for a pentagon and a nonagon.



17 Geometry for star circles.



18 Twist circles for a pentagon and a nonagon.



19 Geometry for twist circles.

$\beta + \delta = \pi$, so $\delta = \pi - (2\beta) = 2\pi/n$. From this, we can find the length of the long side of the orange triangle as

$$t = s\tan(\delta) = R\sin(\pi/n)\tan(2\pi/n)$$

Star circles

Now let's put a circle into each point of Figure 14, as Figure 16 shows. Our goal is to find the radius of the little inscribed circle.

Figure 17 shows the geometry. Let's start with the green triangle. We can see that $\beta = \pi - (\pi/n) = (\pi/2 - \pi/n)$. Moving to the pink triangle, we can see that this also has a right angle and an angle of π/n, like the green triangle, so the far angle (the one touching the center of the circle) must also be β. We can see that $\tan(\pi/n) = h/R$, so $h = R\tan(\pi/n)$.

We're almost there. In the blue triangle, we see that $\alpha = (\pi/2) - \beta = \pi/n$. Thus

$$r = h\sin\alpha = \mathrm{R}\tan(\pi/n)\sin(\pi/n)$$

Placing the circle only requires moving a distance r along the line from the center of the n-gon through the midpoint of its edge.

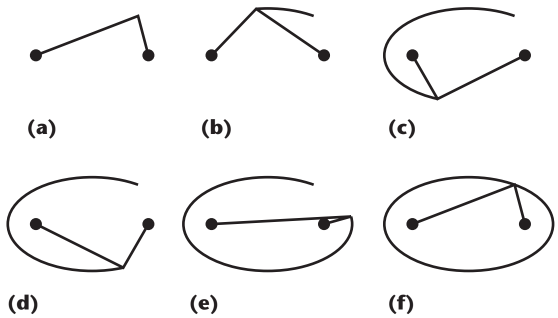
Twist circles

Let's come up with another pattern from circles and regular n-gons. Suppose we take an n-gon, copy it, and give it a twist and a scale so that the midpoints of the new version are lined up with the vertices of the old one. Figure 18 shows the idea. Then we can put little circles into the newly formed triangles. What's their radius?

Look at the orange triangle in Figure 19a. As before, R is the distance from the center of the n-gon to any vertex, π/n is the central angle of the marked polygon, and β is the third angle of the triangle. In Figure 19b we can see that $a = R\sin(\pi/n)$, and since two angles of the blue triangle are π/n and a right angle, the third angle is also β. Isolating this triangle in Figure 19c, we can double it up in Figure 19d to represent the little triangle that's created by the twist and scale operations.

The circle touches the bottom of this triangle, and the upper-right leg as shown. The line from the right vertex to the center of the circle thus splits the angle π/n into two equal pieces, creating the yellow triangle. From this we can read

$$r = \alpha\tan(\pi/2n) = R\sin(\pi/n)\tan(\pi/2n)$$



20 **To draw an ellipse, put down two pegs and tie a piece of string to them. Pull the string taut and move your pen.**

Ellipses

Today's crop formations are mostly composed of straight lines and circular arcs. This makes good sense: these shapes are easy to make with string and posts.

But there's at least one other shape that's almost as easy to make, yet I've rarely seen any in the hundreds of photos of crop circles that I looked at while working on this project: the ellipse.

You might recall that you can draw an ellipse with two pegs and a piece of string (now doesn't that sound appealing for crop circles?). Just put down the pegs, tie the string to each one, pull it taut, put the pen at the tip of the pulled string, and start to move it around the pegs, as in Figure 20. By the time you get back to the starting point, you'll have an ellipse.

This is a nearly perfect crop-circle technique. The two pegs, or *foci*, can be either posts in the ground or friends holding ends of a piece of string. Facing them, and keeping the line taut, you simply walk sideways until you've made a complete circuit, creating the outline of the ellipse as you go.

There are two popular ways to describe the shape of a given ellipse. The form we've just seen, which I call the *circlemaker's ellipse*, can be written (P, Q, s), specifying the locations of the pegs at points P and Q, and the length of string s. The other form of the ellipse, which I'll call the *geometer's ellipse*, is written (h, v, C, θ), where h and v refer to the half-width and half-height of the ellipse, C is the center point, and θ is the counterclockwise angle by which the ellipse is rotated. These parameters are shown in Figure 21.

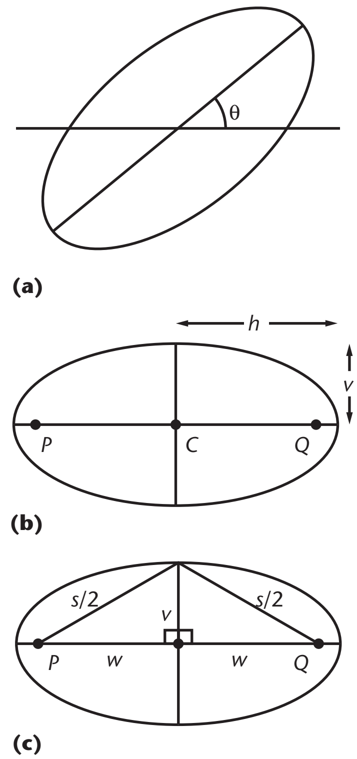
How do we convert from one to the other? Here's how to go from the circlemaker's parameters to the geometer's parameters:

1. Find $\theta = a \tan 2(Q_y - P_y, Q_x - P_x)$.
2. Find $C = (P + Q)/2$.
3. Find $w = |P - Q|/2$. This is the distance $|CP| = |CQ|$.
4. Find h. The following steps don't depend on orientation, so for convenience I'll use the ellipse aligned with the axes in Figure 21b. If we draw the string of length s taut along the $+X$ axis, then it will extend $(h - Q)$ to the right of Q, and $|CP| + h$ to the right of P. Point $Q = (w, 0)$ and $P = (w, 0)$. So adding these together, $(h - w) + (h + 2) = 2h$. Thus $s = 2h$, or $h = s/2$.
5. In Figure 21c, when the string is taut at $+Y$, we have two equal triangles. From either one, and using $h = s/2$, $v = \sqrt{h^2 - v^2}$.
6. Our ellipse is (h, v, C, θ).

In step 1, I used the function $a \tan 2$, which is the standard math library function for computing the arctangent of y/x in the correct quadrant (that is, making sure the result has the right sign). To go the other way, we use this procedure:

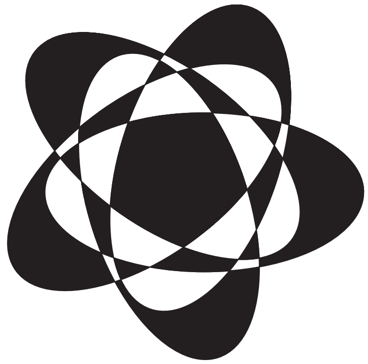
1. From the previous, $s = 2h$.
2. Also from above, $w = \sqrt{h^2 - v^2}$.
3. Compute vector $\mathbf{M} = w(\cos(\theta), \sin(\theta))$.
4. $P = C - \mathbf{M}$, $Q = C + \mathbf{M}$.
5. Our ellipse is (P, Q, s).

Figure 22 shows a formation that I'd like to make that's based on a simple arrangement of ellipses.

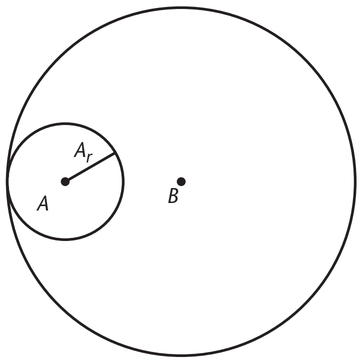


21 **(a) The angle θ gives us the rotation of the ellipse. (b) Geometry of an ellipse. (c) Geometry for converting between the circlemaker's description and the geometer's description.**

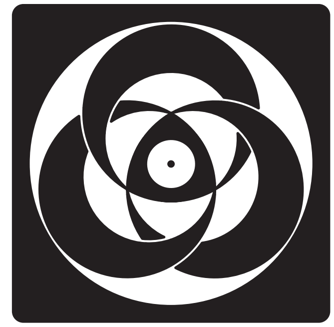
22 **Original design based on ellipses.**



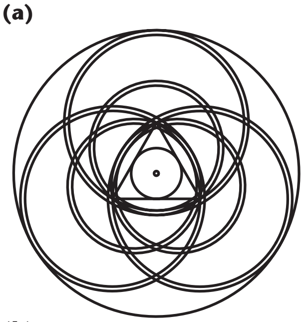
23 **Ellipse-based design inspired by the five-Venn elliptical diagram by Branko Grünbaum.**



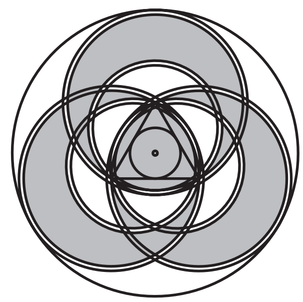
24 **We're given circle A with center *A* and radius A_r, and asked to draw circle B with center *B*, and a radius that makes it tangent to the far edge of A. This is easy on paper, but hard in the field.**



(a)



(b)



(c)

25 **(a) *Folly Barn 2001* formation. (b) Schematic of the formation. (c) The schematic and formation overlaid.**

Another beautiful formation could be based on the five mutually intersecting ellipses discovered by Branko Grünbaum for representing a five-element Venn diagram, as I discussed in my July 2003 column. Figure 23 shows a potential crop formation based on that pattern.

Reconstruction

Now that we've looked at some of the basic geometric tools behind crop circles, let's think about designing formations that we can actually build in the field.

As I discussed before, in the field you can't erase your construction marks. So if you cook up a design that requires you to draw a bunch of lines and arcs to simply locate a point to be used later, try to incorporate them into your design.

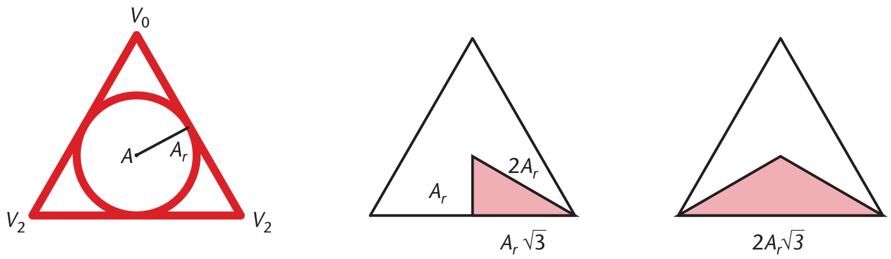
Some other compass-and-ruler techniques are harder to carry out in the field than on paper. For example, we might have a diagram like Figure 24, where we have a circle **A** and a point *B*, and we want to create a new circle **B** that's centered at *B* but is tangent to the far side of **A**. Using hand tools on a paper design we can easily adjust our compass to the right radius by eye, and then draw the new circle. However, this would be difficult to do in the field, because when you're out there in the grain it can be hard to know just where to stand to find that point of tangency. Also, as you walked around to locate that point, you would certainly bend and snap the grain as you wandered.

If you have a design that calls for this kind of step during the paper construction, it's important to work out an efficient, alternative method with which to locate the key points in the field. One nice aspect of working out the design on paper first is that you can find quantitative measures for all sorts of things, and then use those measures in the field.

One of the best ways to learn how to do something is to reverse-engineer high-quality examples that other people have created. Zef Damen has created dozens of careful ruler-and-compass style constructions based on actual formations in the field. He starts with photographs, measures them, and then ultimately checks his reconstruction by overlaying it on the original photo (see the "Further Reading" sidebar for a pointer to his site).

To get the flavor for how large formations get created, let's work through a reconstruction. I've chosen the *Folly Barn 2001* construction (see Figure 25a) because I think it's beautiful, elegant, and about the right level of complexity for us here (crop circles are typically named for the location and year in which they're found). Of course, we can find simpler formations on the crop circle sites, as well as many that are far more complex. I'll paraphrase Damen's analysis here, changing it a bit to make it simpler and easier to follow. That will also make it easier to build the bridge to my description language, which I'll discuss next time.

Our goal is to build the schematic diagram of Figure 25b, which contains all the edges of the darkened regions that we'll flatten with our stompers.

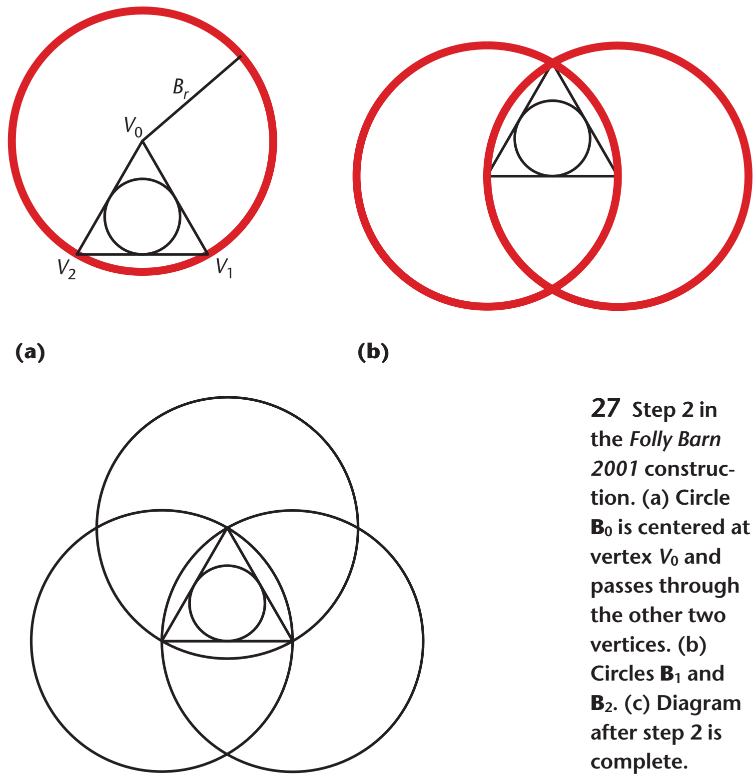


(a) **(b)** **(c)**

26 **Step 1 in the *Folly Barn 2001* construction. (a) Draw circle A and its circumscribed triangle T. (b) Geometry of the lower-right corner of the triangle. It has a radius of $2A_r$. (c) Geometry of the bottom of the triangle; it has a side length of $2A\sqrt{7}$.**

1. We'll start with a circle **A** of radius A_r, and circumscribe an equilateral triangle **T** around it, as in Figure 26a. We can see from the triangles in Figure 26b and c that the distance from the center of the circle to any vertex of the triangle is $2A_r$, and the side length of the triangle is $2A_r\sqrt{3}$.
2. Draw circles $\mathbf{B}_0$, $\mathbf{B}_1$, and $\mathbf{B}_2$, each from one vertex through the other two, as in Figure 27. Notice that we're starting out with a construction that mirrors Hawkins' Theorem 1. Since we know the length of the triangle side from step 1, $B_r = 2A_r\sqrt{3}$.
3. Locate point C_0 at the top of circle **A**, as in Figure 28a (next page). You can find this point by drawing a line from the center of circle **A** to vertex V_0 of triangle **T** and noting where it crosses circle **A**. The coordinates of this point are $C_0 = (0, A_r)$. We want to draw a circle from point C_0 that passes through each of the other vertices of the triangle **T**. Consider the vertex V_0 in the bottom right. As we know from step 1, its coordinates are $V_0 = (A_r\sqrt{3}, -A_r)$. Therefore the distance between these two points is

$$\begin{aligned}|C_0V_0| &= \sqrt{(A_r\sqrt{3})^2 + (A_r - (A_r))^2}\\ &= \sqrt{3A_r^{\,2} + 4A_r^{\,2}} = \sqrt{7A_r^{\,2}}\\ &= A_r\sqrt{7}\end{aligned}$$



(a) **(b)** **(c)**

27 **Step 2 in the *Folly Barn 2001* construction. (a) Circle $\mathbf{B}_0$ is centered at vertex V_0 and passes through the other two vertices. (b) Circles $\mathbf{B}_1$ and $\mathbf{B}_2$. (c) Diagram after step 2 is complete.**

So our new circle $\mathbf{C}_0$ has center C_0 and radius $A_r\sqrt{7}$, as in Figure 28a. Now repeat this for the other two vertices, by drawing a line from the center of **A** to each vertex, locating where it crosses **A**, and drawing a circle of radius $A_r\sqrt{7}$, as in Figure 28b. We have $C_r = A_r\sqrt{7}$.

4. We'll now draw three new circles **D** on the same centers as the circles **B**, but with a radius so that they're tangent to the circles **C** we drew in step 3, as in Figure 29a.

To find the radius of these circles, which I'll call **D**, consider Figure 29b, where I'm showing circles $\mathbf{B}_0$ and $\mathbf{C}_0$ involved in constructing $\mathbf{D}_0$. Circle $\mathbf{B}_0$ is centered at vertex $V_0 = (0, 2A_r)$. The point W that's at the bottom of the circle $\mathbf{C}_0$ is at $W = A_r - C_r = (0, A_r\sqrt{7})$. Thus the distance between them is

$$\begin{aligned}|WV_0| &= \sqrt{0^2 + (2A_r - A_r(1-\sqrt{7}))^2}\\ &= 2A_r - A_r\sqrt{7} = A_r(1+\sqrt{7})\end{aligned}$$

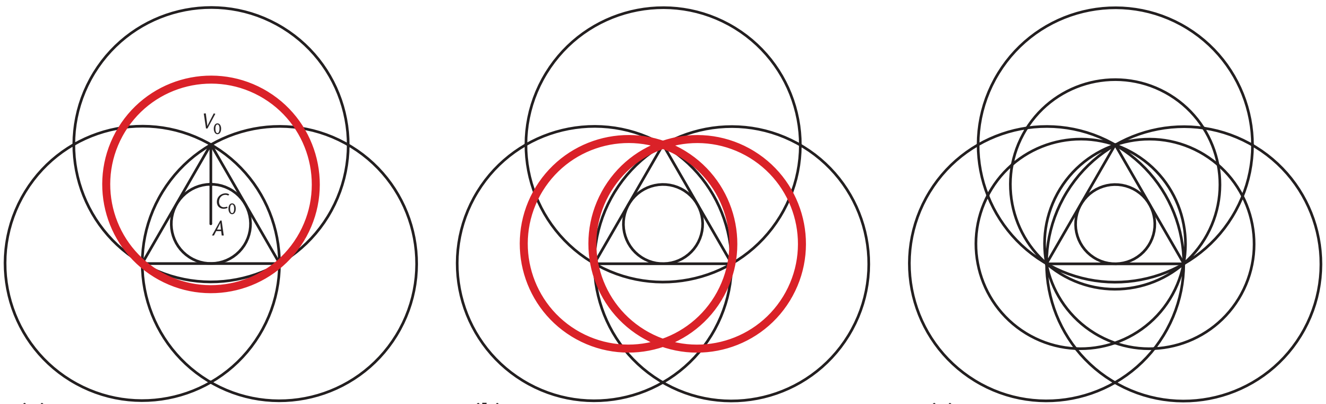
So our three circles **D**, shown in Figure 29d, are centered at the vertices of triangle **T**, and have a radius $D_r = A_r(1+\sqrt{7})$.

5. Now we'll draw a circle **E** from the center of the formation with a radius that makes it tangent to the circles we drew in step 4. Again, finding the radius of this circle now will save time in the field. As we can see from Figure 30, the radius is simply the distance from the center to vertex V_0, which we know is $2A_r$, plus the radius of circle $\mathbf{D}_0$, which we know is $D_r = A_r(1+\sqrt{7})$.

Thus $E_r = 2A_r + D_r = A_r(3+\sqrt{7})$.

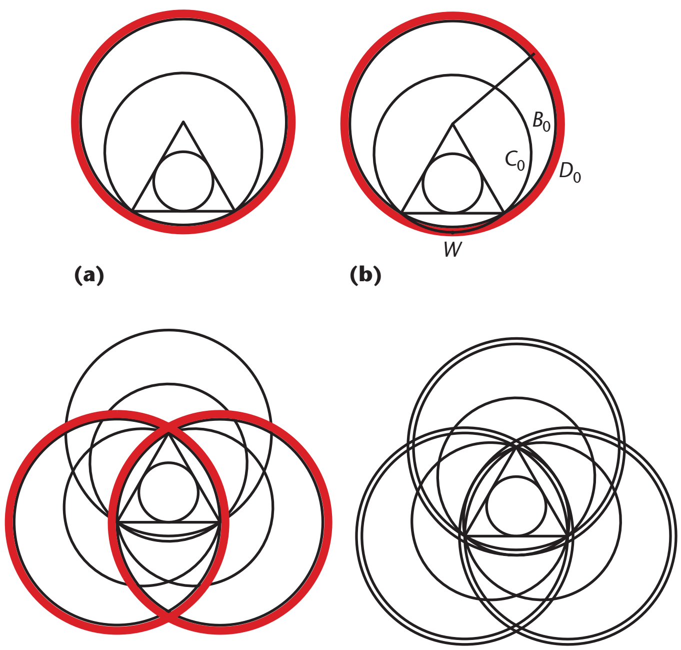
6. Time to draw three more circles! These circles, which I'll call **F**, will be centered on the same points as the circles **C**, but will have a slightly smaller radius.

Figure 31a shows the idea. We want to find the radius of a circle centered at point C_0 that is tangent to circle $\mathbf{B}_0$. Recall that $B_0 = (2A_r, 0)$. The point we want to just touch is found by going up to the center of B_0, and then down by the radius B_r, so $Q = (0, 2A_r - B_r)$. The distance between them is



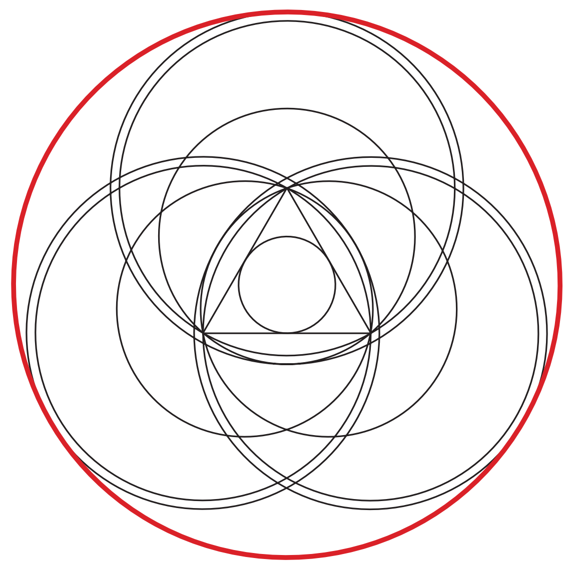
(a) (b) (c)

28 **Step 3 in the *Folly Barn 2001* construction. (a) Circle C₀ is centered at the point of intersection of circle A and the line from the center of circle A to vertex V_0. The circle's radius is set so that it passes through the far two vertices. (b) Circles C_1 and C_2. (c) Diagram after step 3 is complete.**



(a) (b) (c) (d)

29 **Step 4 in the *Folly Barn 2001* construction. (a) Circle D_0 is centered at V_0, and has a radius so that it's tangent to the bottom of C_0. (b) Point *W* is the point of common tangency of D_0 and C_0. (c) Circles D_1 and D_2. (d) Diagram after step 4 is complete.**



30 **Step 5 in the *Folly Barn 2001* construction. Circle E is centered at the center of circle A, and is tangent to the outer edge of circle D_0.**

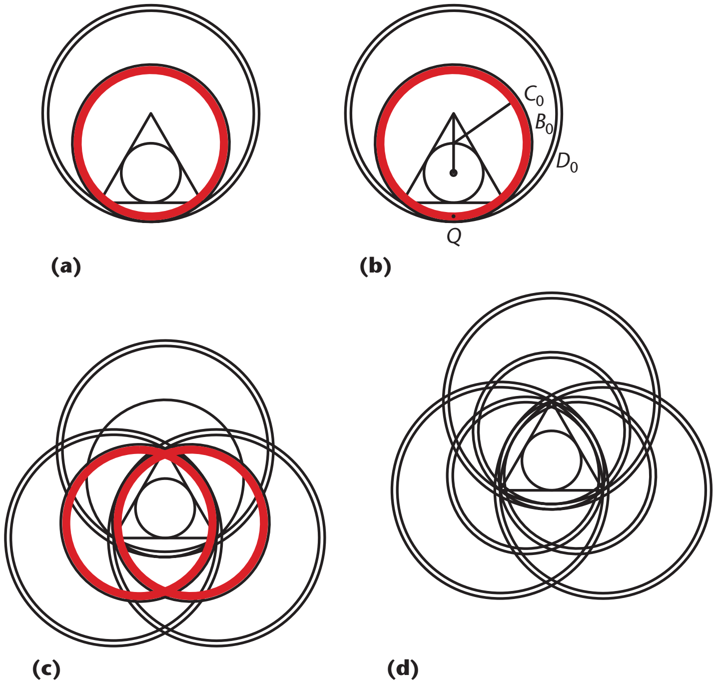
$$\left|C_0Q\right| = \sqrt{(A_r - 0)^2 + (2A_r - B_r)^2}$$
$$= A_r + B_r - 2A_r = B_r - A_r$$

As shown in Figure 31c, we can simply move to the centers of circles **C** and draw circles with radius $F_r = B_r - A_r$.

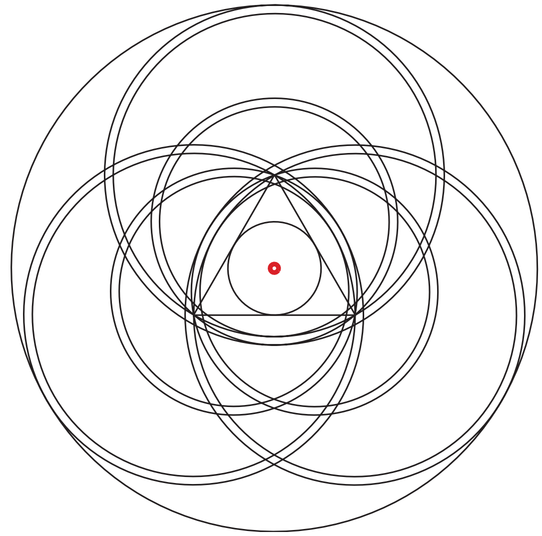
7. We're just about done, but we need to account for the little dot that's at the center. Its diameter is equal to the gap between circles from the last step, so its radius is half that: $G_r = (C_r - F_r)/2$. Figure 32 shows this final circle added in.

We're finished! Our construction in Figure 32 matches the schematic in Figure 25. If we wanted to go out and make this figure, we'd pick a value for A_r, and then we could write down numerical lengths for every radius involved.

In this process I followed Damen's philosophy of determining everything using strictly geometric information. However, we could have taken a number of shortcuts if we weren't feeling so pure. For example, several steps involved computing concentric circles of slightly different radii, like circles **C** and **F**. We could have simply said at the outset that the gap between these two circles should be something like two feet. Then when we'd found the larger circle in the field, we could



31 **Step 6 in the *Folly Barn 2001* construction. (a) Circle $\mathbf{F_0}$. (b) Circle $\mathbf{F_0}$ is centered at C_0, and has a radius that makes it tangent to the bottom of circle $\mathbf{B_0}$ at the point Q. (d) Diagram after step 6 is complete.**



32 **Step 7 in the *Folly Barn 2001* construction. Little circle G is in the center.**

just shorten up the tape by two feet and draw the next one. It would also make our calculations easier on paper.

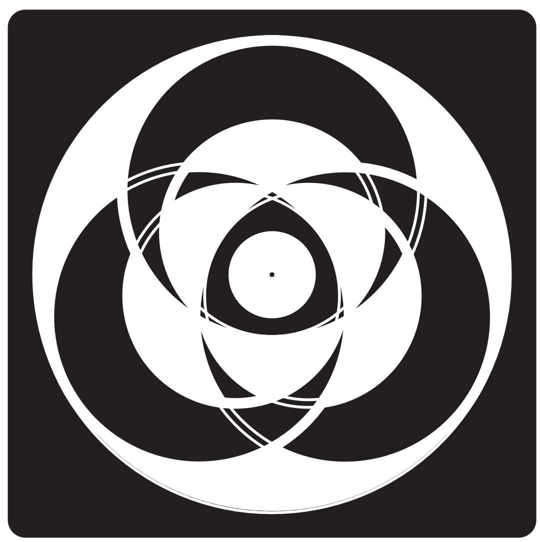
Notice that although I've been talking about marking out complete circles, in this case you wouldn't want to do that. If you stepped through all of the circles we created here in the field, you'd end up cutting extraneous circular arcs through areas that you'd prefer leaving as solid pieces, as Figure 33 shows. While working in the field, you need to keep in mind where you need to stop marking your construction lines and err on the conservative side. When the basic pieces of the design are in place, you can return to incomplete arcs and finish them off, since you'll now know where to stop.

Next time

The geometrical process described above is interesting, but it's no way to produce an actual crop formation in the field.

Next time I'll discuss a small language I've designed, called *Crop*, to help us describe crop formations efficiently, and automatically produce instructions for making them. I'll also describe my experiences creating an original crop circle. ■

Readers may contact Andrew Glassner at andrew@glassner.com.



33 **If we lay down all the marks of Figure 32 in the field, we'll end up cutting through regions that should be left untouched.**