Frieze Groups

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A man goes into a bar and starts drinking hard. The night barman leans over to him and says, "You look sad. . . . I can see you're alone." The man shakes his head and sips at his beer, saying, "It's those darn nuclear freeze groups."

Oh, wait, I got that wrong. I'll start again.

A man gets into his car and starts thinking hard. His wife Carmen leans over to him and says, "You look sad. . . . Did you lose the cellular phone?" The man shakes his head and shifts into gear, saying, "It's those darn unclear frieze groups."

There's no reason to get harried over frieze groups. They're only seven in number, not too hard to understand, and pretty useful. The idea is inspired by friezes—decorative horizontal bands often used in architecture. Figure 1 shows a piece of a story-telling frieze from ancient Egypt. This example shows written language and people at work, but often friezes are simply abstract designs used purely for decoration. Ornamental bands also show up in such diverse crafts as pottery, embroidery, and furniture. When these bands are created by some repeating pattern or motif, and particularly if that pattern has some internal symmetry, then there's a good chance the structure of the design can be represented mathematically as a frieze group.

The value of such a description in graphics is twofold. First, you can generate lots of texture by stamping out little textured polygons in the right positions and orientations. Second, you can go the other way and, given a point on a polygon, find out what the texture value ought to be. In other words, because the pattern is algorithmic, we can synthesize it and generate texture (such as when scan converting) or analyze it to find the texture coordinates we need (such as when ray tracing).

First, imagine the components

The basic idea behind frieze groups is to imagine that you have a design that's printed on a rectangular glass tile. The tile attaches to a metal rod through a fixture at its base, and you want to place copies of the tile along the rod to make a long flat strip. You can't cut or bend the glass, but you can rotate it around the rod, flip it over left-to-right, and move it horizontally along the bar, as shown in Figure 2.

We will restrict our attention to flat bands. In this context, you can rotate the tiles to change their position and orientation, but they must end up in the plane of the page. If you ask yourself how many different kinds of periodic patterns you can generate from these four tiles, you're thinking about frieze groups. You may be surprised that there are only seven fundamentally different patterns.

If you remove the rod and generalize the process to the whole 2D plane, then you're creating wallpaper—designs that repeat smoothly across the plane. There are only seventeen ways of doing this. Wallpaper patterns are very important in computer graphics, because we often create large textured surfaces by taking a little piece of texture and repeating it over and over. I'll leave the more complicated subject of wallpaper patterns for a future column.

The study of frieze patterns is traditionally associated with group theory, a branch of abstract mathematics. In fact, frieze and wallpaper patterns appear almost universally in books on group theory because they are perfect illustrations of those ideas. The group-theory approach to our subject is deeply satisfying because it is very elegant and presents a beautiful chain of thinking that makes the results seem inevitable and certain.

But in this column, I'll take a very informal and visual approach to the subject. I hope you'll believe my arguments, but I'll appeal to your intuition. Intuitive mathematics is no substitute for rigor—some seemingly reasonable conjectures can be completely wrong. Developing your intuition, however, is a great way to approach a subject for the first time, since you can get the big picture up front and fill in details later if you care to. If you find the subject intriguing, dip into the books in the "Further reading" sidebar and investigate the beautiful theory behind these symmetrical patterns.

The infinite band

Our goal is to categorize repeating patterns that lie within an infinite 2D band. More precisely, suppose that you have a pair of infinite parallel lines; the region of the plane between them is where we will draw our pattern. (I'll talk of the pattern running horizontally, but of course the strip can be oriented in any direction.) The centerline of the band is the "rod" upon which we will hang our tiles.

Because the overall pattern repeats, there must be some piece of the pattern that we can isolate, and then use as a rubber stamp. In Figure 3a, I show a simple band pattern and the *fundamental cell*—that little region that contains the essence of the figure. If we made a rubber stamp of the fundamental cell, we could make the pattern just by stamping out images left and right into infinity. The result of translating this entire, infinite band by the width of one cell would give us a new band that is indistinguishable from the original.

1 A drawing of a piece of a 4,000-year-old kiosk of King Sesoris I. The coloring is not historically motivated.

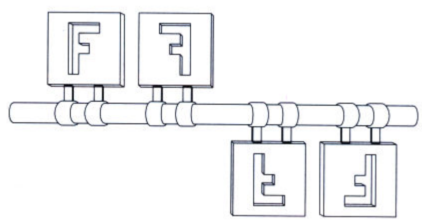
We say that *translation* is an *isometry* of such a band. An isometry is a transformation that preserves shape (isos = equal, métron = measure). For convenience, we will give each isometry its own letter; translation is simply T (unfortunately, the literature contains conflicting sets of notations for the isometries and the patterns they produce; my choice here is motivated by simplicity).

Our goal will first be to find all the possible isometries that an infinite band might possibly contain, then find all the unique combinations of those isometries.

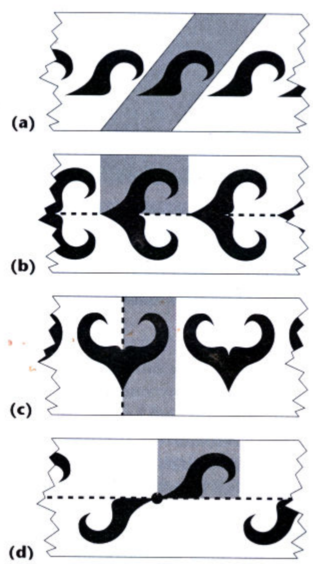
Hall of mirrors

All repeating strip patterns have a fundamental cell that generates the strip by translation. If the entire band is shifted over in either direction by the width of this cell, the result is indistinguishable from the original band. So translation, T, is an isometry of all infinite, periodic bands. What other isometries might an infinite, periodic band contain?

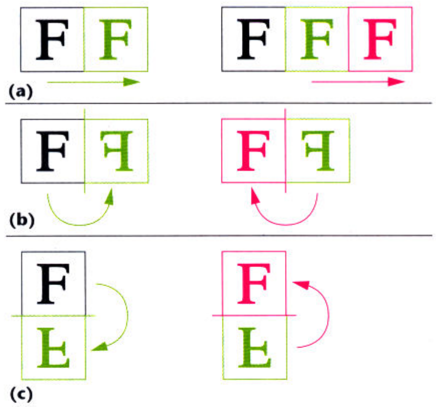
Suppose we look at the centerline along the band and reflect every point above the line to its mirror position below, and vice-versa for those points below the line, as in Figure 3b, and come up with a pattern indistinguishable from the original. Because the mirror line runs hor-



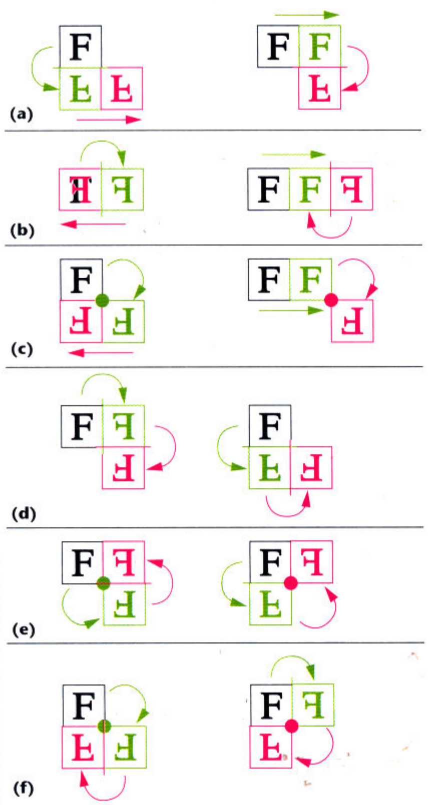
2 Tiles on a rod. If the tiles are to be in the plane of the page, and on the rod, these four orientations are the only possibilities.



3 Bands generated by different operations; the gray zone is a fundamental cell. (a) Translation along the band. (b) Translation and horizontal reflection; the reflection axis is shown by a dashed line. (c) Translation and vertical reflection; the reflection axis is shown by a dashed line. (d) Translation and rotation; the point of rotation is shown by the solid circle, and the axis of translation is the horizontal dashed line.



4 Pairs of repeated transformations. (a) Translation (T) repeated twice. (b) Vertical reflection (V) repeated twice. (c) Horizontal reflection (H) repeated twice. The original tile is black, the result of the first operation is green, and the second result is magenta.



5 Pairs of different transformations. The original tile is black, the result of the first operation is green, and the second result is magenta. A rotation is represented by a circle at the pivot point, and a reflection by a thick line at the mirror.
(a) TH=G and HT=G. (b) TV=X and VT=V. (c) TR=R and RT=R. (d) HV=R and VH=R. (e) HR=V and RH=V. (f) VR=H and RV=H.

izontally, such a band has *horizontal reflection* symmetry, which we denote with the letter H.

Now suppose that we place a reflection line perpendicular to the centerline, in the plane of the band, and exchange the position of points to the left and right of that line. Suppose the result is just like the original, as in Figure 3c. Since the mirror lines runs vertically, we say it has *vertical reflection* symmetry, which we describe with the letter V. The placement of the symmetry line can't be just anywhere—typically it must be right on the border of a copy of the fundamental cell.

Another way to think about these two reflections is as rotations out of the plane of the band. H symmetry spins the band around like a very wide paddlewheel on an old steamboat. V symmetry spins the band like a giant propeller, the infinite ends slicing through space.

Another type of symmetry that the entire band can share is rotational symmetry within its own plane. Pick a point on the centerline of the band and imagine rotating the band around that point. If the band is going to have a chance of being unaffected by the operation, it had better spin by some multiple of 180 degrees. A 360-degree rotation brings it right back to where it started. So the interesting rotation is 180 degrees, as shown in Figure 3d. If the result of a 180-degree rotation is to leave the band looking the same as when it started, then we say it has *half-turn rotational symmetry*, which we denote R.

These four operations—T, H, V, and R—seem like a pretty complete list. Think about it in terms of *fixed points*: those points that don't move under the isometry. For translation, everything moves—there are no fixed points. For both types of reflection, a single line of points remains stationary—the points along the line of reflection. And for rotation, only a single point stays still—the center of rotation. So our four isometries give us no fixed points, a line of fixed points (twice), and a single fixed point. Most of the likely sets of possible fixed points seem accounted for.

This does seem like a complete list. But even in a casual study, we want to make sure we cover all the bases. Can there be any other simple isometries of the band that we haven't considered? It's possible that some combinations of these four isometries could make a new isometry that might not be obvious. Let's try combining them and see what happens.

The mating game

How many useful ways are there to combine our list of four isometries? Let's try applying two isometries, one after the other. We'll read them right to left, like nested trig operations. So HT means to apply translation, then horizontal reflection. If the result leaves us where we started, then it's just as though we had done nothing at all. We represent this by saying that the combination is equivalent to the *identity isometry*, written I. The collection of all isometries that preserve the band is called the *symmetry group* of the band; we say that the band is *invariant* under these operations.

First we'll try each operation twice, as shown in Figure 4. Applying T two times gives us more translation, so there's nothing new there. Applying H twice leaves us right where we started—there aren't any new isometries to be found there. Similarly, doubling up V or R leaves us where we started, so there's nothing new hiding in those combinations, either. In our notation, we can summarize these results as $T^2=T$, and $H^2=V^2=R^2=I$.

Note that this notation doesn't capture everything about these particular isometries. For example, when we write $T^2=T$, we only mean that the equivalent of two translations is one translation—we're not saying how long each translation is. This is to avoid cluttering everything with additional notation that isn't really useful.

How about the other combinations? There are six other pairs, each in two forms, as illustrated in Figure 5. Most of the combinations reduce to one of the simpler isometries. For example, consider HR. First we rotate the cell, then we flip it over the centerline. The result is just like applying V directly. There are only two types of odd ducks in the flock. The first is TV. When we flip the cell across the vertical line, the direction of translation flips with it—thus the translation stomps the tile (in a new orientation) on top of its original image. This is not a new kind of symmetry!

The other odd combination is H and T. Consider HT. First we translate the cell, then we reflect it about the centerline of the band. We get the same result with TH. This result isn't like any of the other isometries, and it is the one isometry we missed in the previous section. This combined move-and-flip isometry, called *glide reflection*, is symbolized by G. Applying G twice means we undo the flip, so two G's in sequence give just a double-length translation. G has no fixed points.

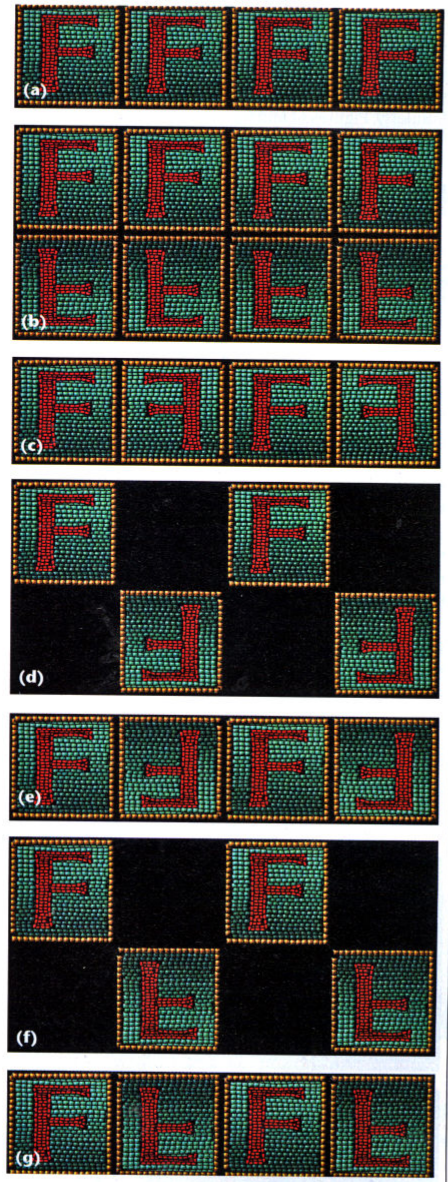
If you think about combining G with the other four isometries, you'll find that nothing new comes of it. One way to see this is to think about writing G as TH or HT: Following it with any other isometry X, you can write X(TH) = (XT)H and then simplify. If that leads to trouble, use HT instead. I know this section hasn't been a proof, but I hope it's suggestive. You can find solid proofs aplenty in the sources listed in the sidebar.

Leading the band

Now we know about all five isometries. Suppose you wanted to make a periodic band pattern containing one or more of these isometries. How many different such patterns could you make? You might think the number would be huge, but it boils down to only seven. Let's see why.

I wanted to use infinite bands to illustrate this part of the article, but then I realized it would require infinite amounts of trees to print them. So instead, we'll just focus on a little piece of the band, and imagine it repeating infinitely left and right. I like to use the letter F in the cell, because it is easy to recognize in any orientation and has no symmetries of its own.

Because we're making a periodic pattern, by definition we know we always need the translation operator T, even if we have nothing else. In fact, translation alone creates the first frieze pattern, which we call F1, as shown in Figure 6a. Now let's add each of our other isometries to T.

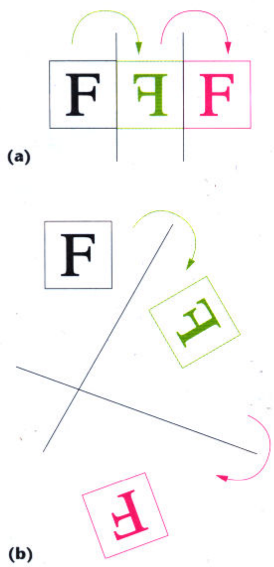


6 (a) [T]. (b) [TH]. (c) [TV]. (d, e) Two bands of the form [TR]. Note that in (d) the center of rotation is at the bottom-right corner of the original tile, while for (e) the center of rotation is in the middle of the right side. (f, g) Two bands of the form [TG].

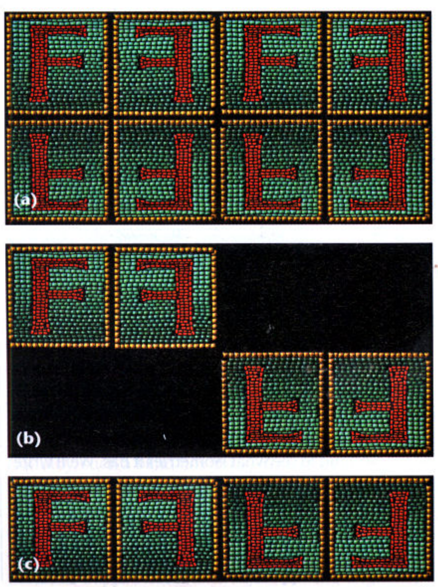
Remember that what we're after now is to analyze the possible symmetries in a band. So we're not applying isometries to a fundamental cell, but rather analyzing the whole band to see what isometries it has. We'll write the symmetry group for the band in square brackets, for example, [TH], to make it clear that for now the order doesn't matter. So F1 is simply [T].

Combining T with each of the other isometries gives us patterns that support [TH], [TV], [TR], and [TG], as shown in Figure 6. That's patterns F2, F3, F4, and F5. Note that Figures 6d and 6e are the same pattern, except that in the latter figure I moved the center of rotation to the middle of the right side of the fundamental cell. Similarly, Figures 6f and 6g are also the same pattern.

The next step is to start combining more than two isometries. We know we always need translation to make a repeating pattern. So we only need to look at combinations of the other four.



7 (a) An even number of reflections in parallel lines is equivalent to translation. (b) An even number of reflections in lines that all meet at one point is equivalent to rotation.



8 (a) [TRH] and [TRV]. (b, c) Two bands of the form [TRG].

These combinations get very simple if you look at them the right way. The trick is to think about successive reflections. Consider Figure 7a, which shows two reflections in parallel lines. The result is just translation, and this is always true for parallel reflections. Figure 7b shows successive reflections in two non-parallel lines. The result is the same as rotation about the point of intersection—you can prove this is always true. So any two reflections sort of cancel each other out: They turn into either translation or rotation, which are handled by their own isometries. So as we build our combinations of isometries, we can only use one of H, V, or G, or we lose them both.

Since we need T, that leaves us with only three ways to combine three isometries: T and R, plus one of the reflections. Figure 8a shows [TRH] and [TRV]—these two symmetry groups describe the same pattern, which is F6. And [TRG] is pattern F7, shown in two forms in Figures 8b and 8c.

We've exhausted the possibilities! Adding in any more isometries will cause them to simply reduce to something we've already covered. Although I certainly haven't presented anything close to rigorous, I hope the basic argument seems reasonable to you. Essentially, we found that an infinite, repeating strip could only have five possible symmetries. Then we found all the ways to combine those symmetries, and discovered that there are only seven different patterns.

Figure 9 offers a set of band patterns, one of each type. The coloration and surface texture is not part of the symmetry pattern; if you want to practice recognizing bands using these examples, focus your attention on the basic shapes of the patterns, not the little details.

Interlocking tiles

A common use of 2D wallpaper patterns is to create interlocking tiles. The humbler, 1D frieze patterns discussed here can be made to interlock as well. The idea is to design tiles that will fit together into the given patterns without creating any overlaps or leaving any gaps.

You can design rules for building interlocking tiles for each of the seven patterns. Basically, you just have to make sure that the edges that touch will fit together and that they don't self-intersect. It's fun to come up with the little geometric instructions for making these interlocking tiles. If you get stuck, take a look at Doris Schattschneider's paper listed in the sidebar—she shows how to build most of the tiles for interlocking frieze groups.

Coding it up

Symmetry programs are great fun to write. The resulting textures can be used to create wood grain on moldings, borders on furniture, hallway carpets, and so on, giving the textures a little more pizzazz than just rubber-stamping the fundamental cell over and over. To create a frieze group image, just pick one of the seven patterns, create the fundamental cell, and replicate it as needed. For interlocking tiles, you can write a program that lets a designer draw curves freehand somewhere on the edge of the tile, then automatically generates the appropriate other curves on other edges.

Further reading

Computer graphics people generally have strong visual skills, so a good way to get into the math is with a book that has lots of good pictures. A book that develops group theory with lots of pictorial illustrations from many different cultures is *Symmetries of Culture* by Dorothy K. Washburn and Donald W. Crowe (University of Washington Press, 1988). Another big book that offers tons of examples of different symmetry patterns is *Handbook of Regular Patterns* by Peter S. Stevens (MIT Press, 1981). A wonderful warehouse of copyright-free examples of many types of patterns can be found in *Decorative Symbols and Motifs for Artists and Craftspeople* by Flinders Petrie (Dover Publications, 1986).

If you want to get into the math, a good visual starting place is *Groups and Their Graphs* by Israel Grossman and Wilhelm Magnus (Random House, 1964). Another visually oriented book, but much heavier on the mathematics, is *Incidence and Symmetry in Design and Architecture* by Jenny A. Baglivo and Jack E. Graver (Cambridge University Press, 1983). The study of isometries and their interactions is called transformational geometry; any good introduction to group theory will give you the tools to till this field. A great introductory paper on color symmetry for frieze groups is "In Black and White: How to Create Perfectly Colored Symmetric Patterns" by Doris Schattschneider (*Computers & Mathematics with Applications,* Vol. 12B, Nos. 3 & 4, 1986, pp. 673-695).

Free-running programs can create eye candy of all sorts, from screen savers to animated backgrounds. Consider creating a long band that winds back and forth across the screen, with a changing pattern replicated across it, and maybe passing through a random series of band types.

There are some fun programming projects associated with frieze groups. You can write each of the isometries as a matrix operation, so that any combination of isometries can be captured by a single composite matrix. You might try writing out the matrices for each of the five isometries T, H, V, R, and G. Then write a program that takes as input a texture tile and generates appropriately transformed versions of that tile to make a band.

Another useful routine takes as input an original tile, a band type, and a point anywhere on the band; the routine analyzes the position of the point to return the appropriate point in the original tile that corresponds to the selected point. For example, if the original tile runs from (0,0) in the lower left to (1,1) in the upper right, the band pattern is [TH], and the input point is (3.2, 0.6), then the routine would return (0.8, 0.6)—the location in the original tile that ends up at the input point after being transformed there.

9 The seven frieze groups.

If you want to get into color, see the further reading sidebar for some useful information on color symmetry.

When you allow repeated reflection lines throughout the plane (rather than the special horizontal and vertical orientations we used here), you can fill the plane with images. Similar beautiful patterns can be seen by looking into a kaleidoscope. The kaleidoscope's class of symmetrical designs can be understood from the point of view of point groups—but that's another column. ■