Circular Reasoning

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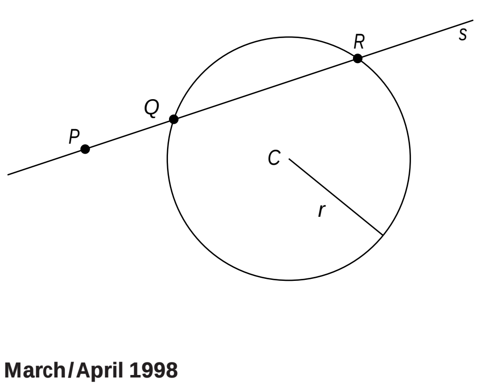
This month I'll talk about two interesting relationships between circles and lines. The first topic is pretty simple but cool, and I suspect there's a computer graphics application out there that can use this to run faster or better. The second topic is Ptolemy's Theorem, which is a generalization of the triangle inequality. I'll show how it can be used to derive the angle-addition formulas (which I always forget). Then I'll extend last month's topic of reflection and show how Ptolemy's Theorem can be used to prove that Snell's Law and Fermat's Principle of Least Time both lead to the same geometry of refraction.

Circular powers

We'll begin with an important property of lines and circles that I haven't seen mentioned before in graphics literature. Figure 1 sets the stage: We have a circle with center C and radius r, a point P not on the circle, and a line s through P that intersects the circle at points Q and R. We'll write $C_r(P)$ for the value of point P in the implicit formula for the circle. I'll demonstrate the remarkable fact that $C_r(P)$ is the product of the distances $|PQ|$ and $|PR|$. In symbols, $C_r(P) = |PQ| \bullet |PR|$.

To see this, we only need to write out the standard intersection of a ray and a circle—this is the familiar algebra that appears in every ray-tracing program. I'll use vector notation for two reasons. First, it's a lot less messy than writing out all the coefficients. Second—well, the second reason is sneaky, as you'll discover in a moment. Writing out the value of the circle at an arbitrary point A, its value $C_r(A)$ may be written in vector notation as

$$\begin{aligned} C_r(A) &= (A - C) \bullet (A - C) - r^2 \\ &= A \bullet A - 2C \bullet A + C \bullet C - r^2 \end{aligned}$$



1 The circle (C,r) and a point P. The line s through P intersects the circle at Q and R.

We'll write the line s as a parameterized ray with origin P and direction vector $\mathbf{V} = (Q - P)$. This means that points on s are given by various values of t in the expression $s = P + \mathbf{V}t$. Plugging this into the circle equation to find $C_r(P + \mathbf{V}t)$ and then gathering terms of t, we find

$$\begin{aligned} 0 &= (P + \mathbf{V}t)\bullet(P + \mathbf{V}t) - 2C\bullet(P + \mathbf{V}t) + (C \bullet C) - r^2 \\ 0 &= t^2(\mathbf{V} \bullet \mathbf{V}) + t[2\mathbf{V} \bullet (P - C)] + [(P - C)\bullet(P - C) - r^2] \\ 0 &= at^2 + bt + c \end{aligned}$$

The roots t_0 and t_1 of this quadratic equation are the values of t, which generate points where the ray intersects the sphere. Let's find the product p of the two roots—this will come in handy in a moment. Writing d for the discriminant $d = \sqrt{b^2 - 4ac}$, we find

$$p = t_0 t_1 = \left(\frac{-b + d}{2a}\right)\left(\frac{-b - d}{2a}\right) = \frac{b^2 - d^2}{4a^2}$$

If we plug in the value for d, expand, and simplify, we find

$$p = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{c}{a}$$

This is a nice general relationship to keep in mind for quadratic equations.

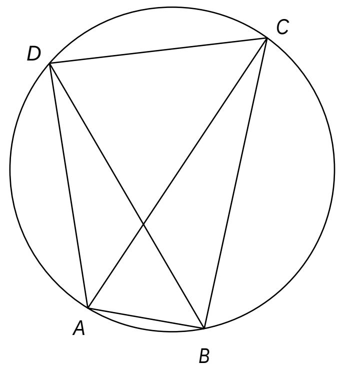
Now back to business. Let's assume that our direction vector $\mathbf{V}$ is normalized, so $\mathbf{V} = (Q - P)/|Q - P|$. This means that the values of t corresponding to the roots of the equation are exactly the same as the distances from P to Q and P to R. That is, if t_0 and t_1 are the two roots (and t_0 is the lesser one), $t_0 = |PQ|$, and $t_1 = |PR|$. Since $\mathbf{V} = 1$, then $a = (\mathbf{V} \bullet \mathbf{V}) = 1$, and thus $p = c/a = c$. Using our value of c from the quadratic formula above,

$$p = t_0 t_1 = c = (P - C) \bullet (P - C) - r^2$$

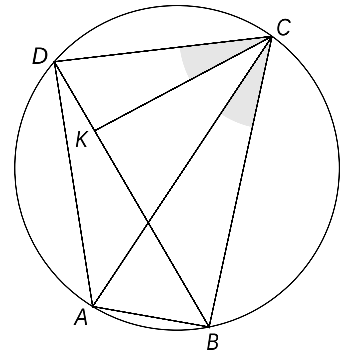
which is simply $C_r(P)$, the value of point P in the circle equation.

So we have proven that the product of the distances is the same as the value of P in the circle's equation—that is, $C_r(P) = |PQ|\bullet|PR|$, as promised.

Notice that nowhere did we actually use the fact that we were in 2D. The vector notation I used doesn't care



2 **A cyclic quadrilateral *ABCD*.**



3 **Point *K* is placed so that $\angle KCD = \angle BCA$.**

how many dimensions we're in. That's the sneaky reason I used vectors—I was working with lines and circles in the plane, but the result is equally true for lines and spheres in space.

Ptolemy's Theorem

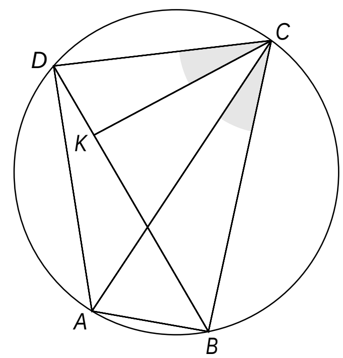
Our second topic this month is Ptolemy's Theorem. Claudius Ptolemy (AD ~87~150) was an astronomer, mathematician, and geographer who lived in Alexandria, Greece. He wrote an early atlas—remarkable for the fact that not only did he describe the places listed, but he also included their latitude and longitude. His other major work was a set of books called *The Mathematical Collection*. This was later translated into Arabic under the title *Al Magiste*, or *The Greatest*. This title was later corrupted, and the books are now often called the Almagest.

The Almagest consisted of 13 books giving early algorithms for 2D geometry as well as an astronomical system for the motion of the stars and planets. The Earth and moon had their orbits centered on the Earth, while everything else revolved around those centers. Such geocentric systems remained popular until the Copernican Revolution in the sixteenth century. Ptolemy also gave an approximate value of π as $377/120 \approx 3.141666$, which is accurate to one part in ten thousand—accurate enough to sail a ship for a few weeks and then find a port by eye.

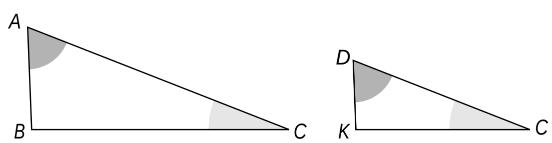
One of the still-influential pieces in the Almagest is a proof of what we now call "Ptolemy's Theorem." This can be considered an extension of the triangle inequality. Recall that the triangle inequality states that for any three points A, B, and C in the plane, $AB + BC \geq AC$. Ptolemy's Theorem gives a similar result for cyclic quadrilaterals.

A cyclic quadrilateral is a figure created by four unique points on a circle. Figure 2 shows a generic cyclic quadrilateral, which I've labeled counterclockwise as *A, B, C, D*. Ptolemy's Theorem says that if you multiply the lengths of opposite sides and add the products, this will equal the product of the diagonals. In fact, equality only holds if the points lie on a circle (just as the triangle inequality is only equal when the points are colinear). We will stick with the cyclic equality version here.

To prove Ptolemy's Theorem, we'll create an additional point K on line DB, such that $\angle KCD = \angle BCA$, as in Figure 3. Without loss of generality, we will assume that K lies closer to D than the intersection of CA and DB. We will find two sets of similar triangles to determine the



4 **Both the inscribed $\angle CAB$ and $\angle CDB$ include arc *BC*. Therefore they're equal.**



5 **Similar triangles ΔCAB and ΔCDK.**

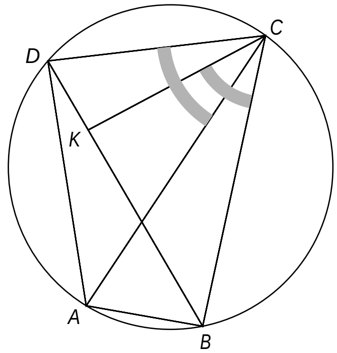
relations that lead to the theorem. We will use unsigned distances, so that $|DB| = |BD|$. Furthermore, to reduce clutter I will leave the vertical bars off of the distance measures. Unless I specifically refer to a pair of letters as something like "arc AB," AB will mean $|AB|$.

First, observe that $\angle CAB$ and $\angle CDB$ are both inscribed angles that include arc BC, as in Figure 4. As we saw last time, this means that these two angles are equal. Thus, as Figure 5 shows, triangles ΔCAB and ΔCDK are similar, since both have two common angles. Because they're similar, we can write

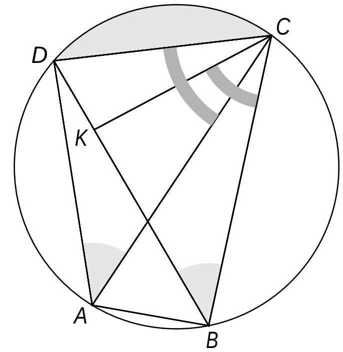
$$\frac{AC}{AB} = \frac{DC}{DK}$$

or $AC \bullet DK = AB \bullet DC = AB \bullet CD$

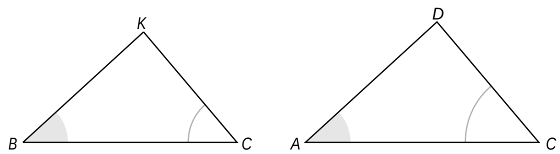
For the second set of triangles, note that $\angle BCA = \angle KCD$ by construction. Adding $\angle ACK$ to both, we have $\angle BCA + \angle ACK = \angle KCD + \angle ACK$, or $\angle BCK = \angle ACD$, as shown in Figure 6. Furthermore, both $\angle DBC$ and $\angle DAC$ are inscribed angles including arc CD, which means they're also equal, as shown in Figure 7. And $\angle KBC = \angle DBC$. Thus triangles ΔBCK and ΔACD are similar. Figure 8 shows them side-by-side. We can write a similar pair of ratios as last time:



6 Equal angles ∠*BCK* and ∠*ACD*.



7 Angles ∠*DBC* and ∠*DAC* are both inscribed and include arc *CD*, so they're equal.



8 Similar triangles △*BCK* and △*ACD*.

$$\frac{BC}{BK} = \frac{AC}{AD}$$

or $AC \bullet BK = BC \bullet AD$. Now we'll add these two equalities together:

$$AC \bullet KD + AC \bullet BK = AB \bullet CD + BC \bullet AD$$
$$AC \bullet (KD + BK) = AB \bullet CD + BC \bullet AD$$
$$AC \bullet DB = AB \bullet CD + BC \bullet AD$$

This last line is Ptolemy's Theorem. As promised earlier, it tells us that for a cyclic quadrilateral, the product of the lengths of the diagonals equals the sum of the products of the lengths of opposite sides.

Angle addition

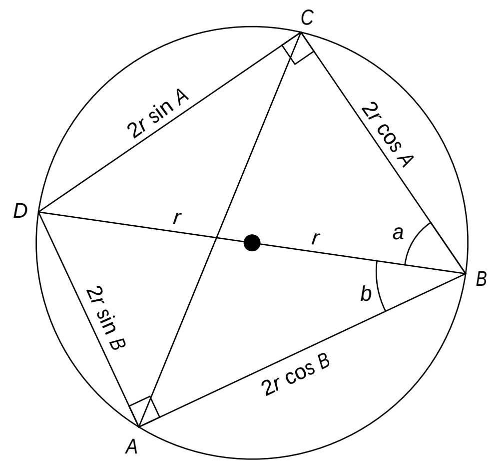
To get some experience with Ptolemy's Theorem, we can derive the formula for finding the sine angle-addition formula $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

We'll start with a cylic quadrilateral of radius r, as shown in Figure 9, with $\alpha = \angle DBC$ and $\beta = \angle ABD$. Points B and D lie at opposite ends of a diameter, thus $BD = 2r$. Recall that the law of sines tells us that for a triangle ΔABC, $a/\sin A = b/\sin B = c/\sin C = 2r$. We use this law to find $AC = 2r \sin B$, where angle $\angle B = \alpha + \beta$. Note that angles A and C are both right angles, because they're inscribed in the semicircles made by diameter BD—thus we can label them with the sines and cosines as shown.

Now we just write out Ptolemy's Theorem, replacing each length with its value from Figure 9, and simplify:

$$AC \bullet DB = AB \bullet CD + BC \bullet AD \; 2r\sin(\alpha+\beta) \bullet 2r$$
$$= 2r\cos\beta \bullet 2r\sin\alpha + 2r\cos\alpha \bullet 2r\sin\beta$$
$$\sin(\alpha+\beta) = \cos\beta\sin\alpha + \cos\alpha\sin\beta$$

which is the standard formula for $\sin(\alpha+\beta)$. I love it when these things work out so nicely. The other variations, for example, $\cos(\alpha-\beta)$, can all be found by plugging in standard trig identities. Now we can move on to a more challenging application.



9 A cyclic quadrilateral where *DB* is a diameter of length 2*r*.

From Snell to Fermat via Ptolemy

Refraction is an important part of our visual world. Every transparent object bends light to some degree as that light passes through it. This phenomenon gives rise to everything from multicolored prisms to the ability of our eyes to focus at different distances.

In January 1998 I wrote about proving the law of mirror reflection by using the mathematical technique of reflection. The two had a lot in common, which probably wasn't too surprising. Now I'll show how to use Ptolemy's Theorem to prove that Snell's Law (an algebraic relationship) leads to Fermat's Principle of Least Time (a physical hypothesis).

The law of specular reflection tells us what happens to a ray of light when it passes from one medium to another. When light passes through the boundary, or interface, from an incident medium i (say, air) to a transmitted medium t (say, glass), its speed v changes from v_i to v_t. This causes the light ray to bend, as illustrated in Figure 10. We won't discuss the mechanics of that bending here; it's covered in detail in most optics and computer graphics texts.

People generally use two laws to compute the transmitted angle θ_t: Fermat's Principle of Least Time and

Snell's Law. We encountered Fermat's Principle in the January 1988 column when we looked at reflection: It says that light takes the least amount of time to get from one point to another. As in that column, we'll assume a world where light travels in straight lines. Suppose the light travels from air point A to surface point S and then to glass point G. Fermat's principle tells us that the time it takes to get from A to S to G must be a minimum. Given A and G, our job is to find S.

Note that S is not simply on the line AG. Remember that we're looking for the least time of flight, which is not necessarily the shortest path. We want to travel less distance where the material is denser, even if it means we travel farther in the more rarified medium.

In terms of Figure 10, we can observe that the time it takes light to get from A to S is the distance divided by the speed in the incident medium, or AS/v_i, and similarly the time from S to G is SG/v_t. Thus we want to minimize $AS/v_i + SG/v_t$. In symbols, for any other $\hat{S}$ on the surface, we want to show that the path $A\hat{S}G$ takes longer than ASG:

$$\frac{A\hat{S}}{v_i} + \frac{\hat{S}G}{v_t} > \frac{AS}{v_i} + \frac{SG}{v_t}$$

A famous geometrical relationship, Snell's law tells us where S is located. In terms of the geometry of Figure 10, Snell's Law says

$$v_t \sin \theta_i = v_i \sin \theta_t$$

I will show that Snell's Law and Fermat's Principle are equivalent. First assume Snell's Law, and then Fermat's Principle is automatically satisfied. Let's see how this works. Figure 11 shows the basic setup. I've drawn a circle (C,r) through points A, S, and G. I've also drawn a vertical line v perpendicular to the horizontal interface h, and found the point B where v intersects the circle (C,r) below line h.

First, we need to find the lengths BA and BG. Note that $\angle ASB$ forms an inscribed angle in the circle. Recall that in the column on specular reflection we showed that an inscribed angle of θ equals a central angle of 2θ. Figure 12 shows how to find the length of a chord UW in a circle (C,r). If V is the midpoint of UVW, then angle $\angle VCW = \theta/2$, and $VW = r \sin(\theta/2)$, so $UW = 2VW = 2r \sin(\theta/2)$. Returning to BA, the inscribed angle $\angle ASB$ is $\pi - \theta_i$, so the central angle is $2(\pi - \theta_i)$, and therefore the length of the chord is

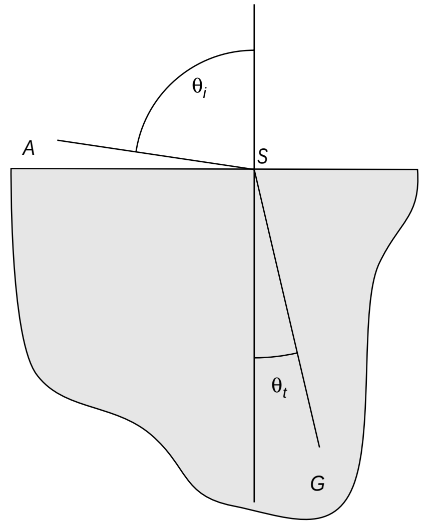
$$BA = 2r \sin(2(\pi - \theta_i)/2) = 2r \sin(\pi - \theta_i) = 2r \sin \theta_i$$

By similar reasoning, $BG = 2r \sin \theta_t$.

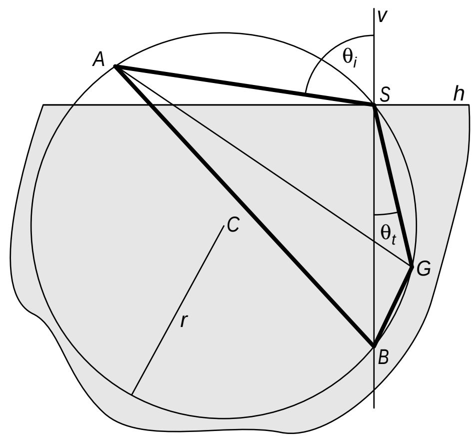
Writing out our chord lengths and using Snell's Law, we find

$$BA = 2r \sin\theta_i = 2r \frac{v_i \sin\theta_t}{v_t} = \frac{k}{v_t}$$

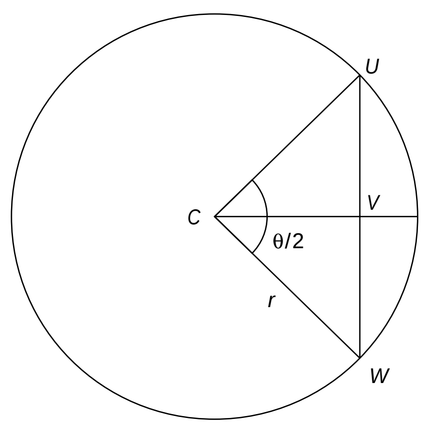
$$BG = 2r \sin\theta_t = 2r \frac{v_t \sin\theta_i}{v_i} = \frac{k}{v_i}$$



10 **The basic geometry of refraction from air to surface to glass.**



11 **The inscribed quadrilateral *BGSA*.**



12 **Finding the length of a chord.**

where $k = 2rv_i \sin \theta_t = 2rv_t \sin \theta_i$.

Now we'll use Ptolemy's Theorem. Since A, S, G, and B all lie on a circle, in that order, we have

$$AG \bullet BS = AS \bullet BG + SG \bullet BA$$

Substituting our lengths for BG and BA from above,

$$AG \bullet BS = AS\frac{k}{v_i} + SG\frac{k}{v_t} = k\left(\frac{AS}{v_i} + \frac{SG}{v_t}\right)$$

This is the critical result. We'll use it in a moment to complete the proof.

For a moment, imagine some other point $\hat{S}$ on the line h between the two media. Is it possible that this point, in violation of Snell's Law, could lead to a shorter time of flight from A to G? Since S and B are both on v, which is perpendicular to h, $B\hat{S} > BS$ for all $\hat{S} \neq S$ on h. Multiplying both sides by AG, we get

$$AG \bullet B\hat{S} > AG \bullet BS$$

Now we use the result from above. Replacing both sides of the inequality with their equivalent formulas from Ptolemy's Theorem and factoring out the common factor k, we find

$$\frac{A\hat{S}}{v_i} + \frac{\hat{S}G}{v_t} > \frac{AS}{v_i} + \frac{SG}{v_t}$$

which was what we had hoped to find. So if we choose S in accordance with Snell's Law, then no other point $\hat{S}$ on the interface h can form a path $A\hat{S}G$ with a shorter time of flight than ASG.

Thus we used Ptolemy's Theorem to show that if we assume Snell's Law, we have also satisfied Fermat's Principle of Least Time for refraction.

Wrapping up

I'm surprised that I haven't seen more on Ptolemy's Theorem in general—it seems like a very useful tool. There are often lots of ways to prove these useful 2D theorems, and after I worked out one to my satisfaction, I hunted around on the Net for something better. I found it! The nice little proof that I used in this column follows the proof given at http://www.cut-the-knot.com/proofs/ptolemy.html. The triple-play using Ptolemy's Theorem for refraction is based on the discussion by Dan Pedoe in *Geometry* (Dover Publications, 1970).

There must be lots of ways to use these bits of circle-and-line geometry to accelerate ray tracing, but I'm not sure how. I'd be happy to hear from any readers who find good applications for these geometrical tidbits.

After I had sent this column to *CG&A* for publication, I mentioned it to Jim Blinn, who remarked that he had created an animated version of Ptolemy's Theoerem and the angle-addition formulas as part of his work on *Project Mathematics!*. He brought in the tape the next day, and it covered these topics in Jim's usual clear and entertaining style. I recommend you keep an eye open for "Sines and Cosines, Part III" the next time *Project Mathematics!* appears on your local television. ■

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