String Crossings

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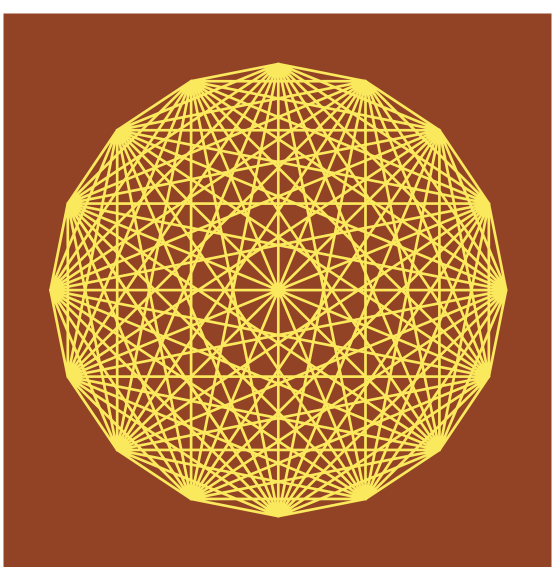
When I was about 10 years old, my local high school held a summer crafts class. Each day we were given something new to do: hammering ashtrays out of metal disks, weaving lanyards, or filling bottles with layers of colored sand. One project involved driving little brass nails into a piece of wood so that they were equally spaced around a circle, and then winding shiny metal wire between every pair of nails. Figure 1 shows the idea. The result was a shiny spider's web of wire that each kid proudly took home as a piece of art.

I don't know what the parents did with those creations, but I do know that the cool pattern stayed in my mind. I didn't know it at the time, but I'd learned an algorithm for making an aesthetic visual piece. Years later, when I started to learn computer graphics, I programmed up these string-art creations on vector storage tubes and plotters—mixing colors and changing line widths—and generally having a fine time.

I hadn't thought about these designs for years, until recently my artist friend Dan Robbins mentioned that he was planning a sculpture that involved a similar kind of design (discussed in the next section). Part of his sculpture involved placing objects at the intersections of pairs of strings. He wanted to plan part of the sculpture in advance and asked me if there was a way to determine how many strings were necessary to get a desired number of intersections.



1 String art from a 16-sided regular polygon.

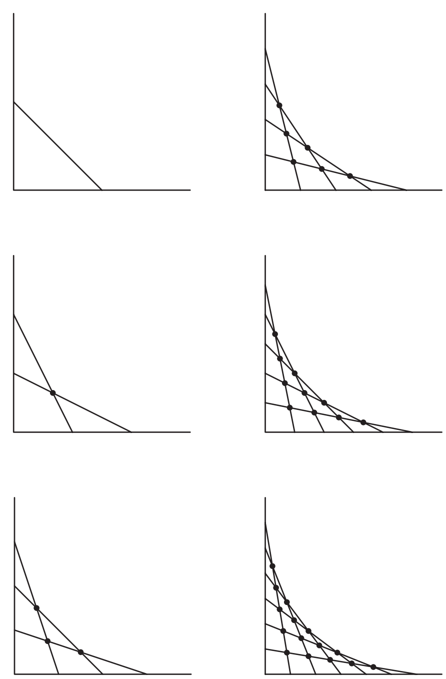
This column starts with an answer to Dan's question. I'll also explore some other questions derived from these simple but elegant little string-art patterns. I discovered that these lovely little geometric patterns lead to numerical patterns every bit as graceful and aesthetically pleasing.

The right angle

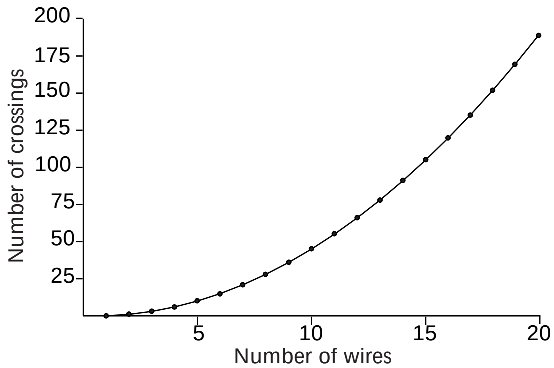
Let's begin with the simplest example. Take two line segments and place them at right angles to each other. Divide each line into $N + 1$ equal pieces. Then run strings from the internal divisions on one line to those on the other. If we number the divisions $1, \ldots, N + 1$ from the common point in the lower left, then the nails would be numbered $1, \ldots, N$. From any nail i on the vertical piece we'll run a string to nail $N - i + 1$ on the horizontal, for a total of N strings. Figure 2 shows the progression for the first few values of N.

We'll begin with a precise form of Dan's question: How many crossings k are there for a given value of N?

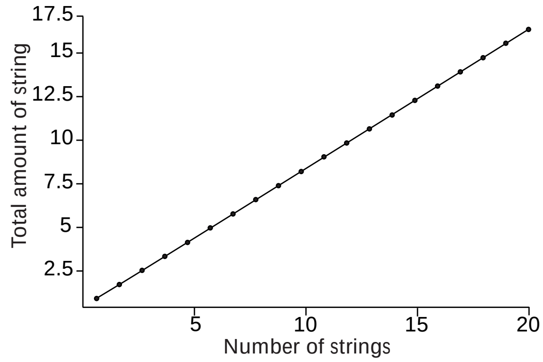
You can see from Figure 2 that when $N = 1$ we have only one string, so clearly $k(1) = 0$. When $N = 2$ we have



2 The right angle string-art patterns for $N = 1$ through 6.



3 **A plot of the data in Table 1.**



4 **A plot of the total amount of string required as a function of the number of individual pieces used.**

only two strings, so $k(2) = 1$. The general pattern may be clear to you from examination. Take any of the patterns, and consider the string running from nail N on the vertical axis to nail 1 on the horizontal axis. That string crosses over all the other strings; since there are N total—and this string is one of them—that yields $N - 1$ crossings. Now the next string also crosses them all, but we've already accounted for one of the strings, so now we have $N - 2$ new crossings. And so it goes, until the last string adds just 1 new crossing and we're done.

So if we have $N + 1$ segments and thus N lines, then the number of crossings is given by $N + (N - 1) + \ldots + 2 + 1$. There's a nice closed-form solution for this sort of summation, rumored to have been discovered by the famous mathematician Carl Friedrich Gauss (1777-1885) as a child.

The story goes that when Gauss was a child, his math teacher came to class unprepared one day. The teacher decided to fill the class time by instructing the students to add up all the numbers from 1 to 100. Most of the students started writing down all the numbers in a big column in preparation for adding them up. But in only a few seconds Gauss announced to the teacher that the answer was 5,050. The teacher assumed that Gauss had simply learned this result as a piece of trivia. So she set him about the more time-consuming task of adding the numbers from 1 to 500. After only a moment's paperwork, Gauss announced the answer was 125,250.

Apocryphal or not, the way to repeat this kind of performance is to note that in a summation from 1 to N, the outermost pair of numbers add up to $1 + N$. Working inwards, the next pair adds up to $2 + (N - 1) = 1 + N$. The next pair is $3 + (N - 2) = 1 + N$, and so on. If N is even, then exactly $N/2$ of these pairs exists. If N is odd, then there are $N/2$ pairs, plus one extra number left alone in the middle: $N/2$ itself. Thus regardless of whether N is even or odd, we can write

$$\sum_{i=1}^{N} i = \frac{N}{2}(N+1)$$

This elegant summation formula is just what we need for our crossing problem. Since we're counting from 1 to N, the sum is $(N/2)(N + 1)$, which is a nice little answer. That's how many intersections you get from N strings. Table 1 gives the value of $k(N)$ for a range of N's, which are plotted in Figure 3. It's no surprise that the curve is a parabola, since $k(N)$ is a quadratic in N.

If you want to create a piece of art with a particular number of crossings, then you can simply solve for N given k using the quadratic formula and taking the positive root:

$$N = \left(1 + \sqrt{1+8k}\right)/2$$

Most of the values of N that result from this process aren't integers.

I wondered how much string it takes to make one of these pieces of string art. Let's assume that the lengths of the two sides are each 1. Then we can write any line as joining coordinate α along one axis with $(1 - \alpha)$ along the other. The length of that line is then simply

$$\sqrt{\alpha^2 + (1-\alpha)^2} = \sqrt{\left(\alpha\sqrt{2} - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}}$$

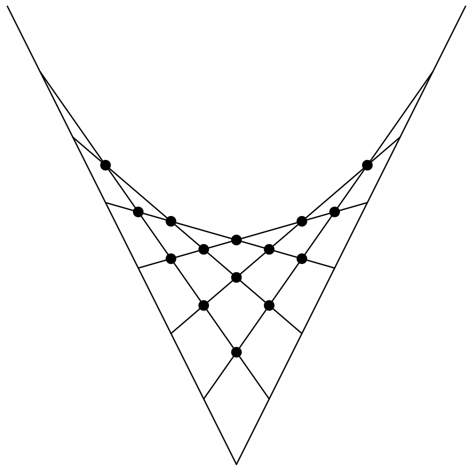
To find the total amount of string, we just march an index i from 1 to N, compute $\alpha = i/(N + 1)$, and add up the lengths:

$$\sum_{i=1}^{N} \sqrt{\left(\frac{i\sqrt{2}}{N+1} - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}}$$

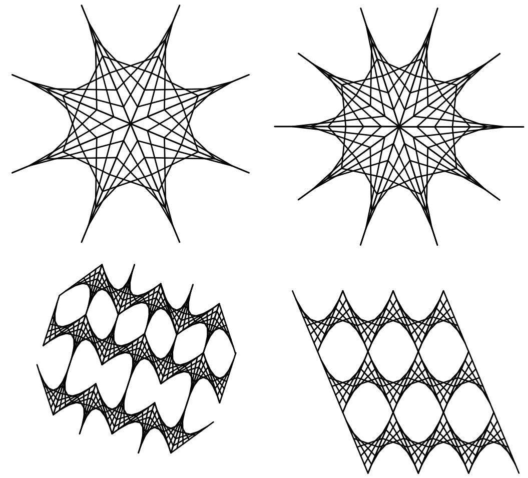
The resulting values are plotted in Figure 4. I was surprised to find that the result is linear. Each time you add a new string, you shorten the others by just the right

Table 1. The number of crossings for a given number of wires in a right-angle pattern.

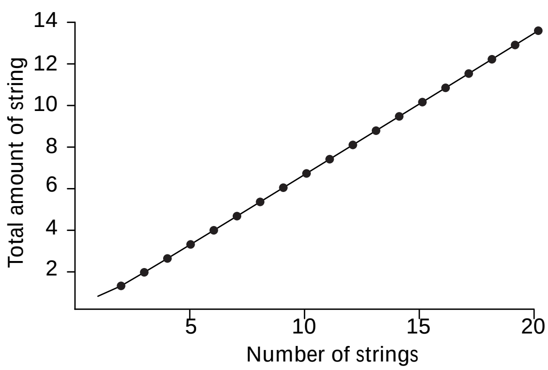
Number of wires	Number of crossings
2	1
3	3
4	6
5	10
6	15
7	21
8	28
9	36
10	45
11	55
12	66
13	78
14	91
15	105
16	120
17	136
18	153
19	171
20	190



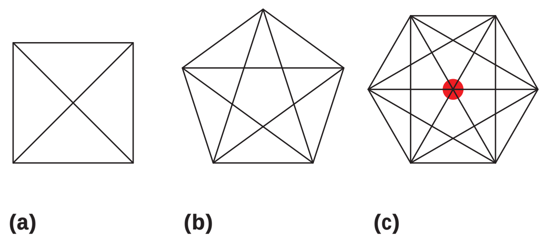
5 **An acute string pattern.**



7 **Some designs made with acute string patterns.**



6 **A plot of the total amount of string required for a 30-degree acute string pattern as a function of the number of individual pieces used.**



8 **Some complete regular graphs. (a) *N* = 4. (b) *N* = 5. (c) *N* = 6. Note the three lines crossing over the red dot.**

amount so that the total amount of new string is a constant. I don't really understand why this should be so.

A closer angle

Let's now close up the 90-degree angle into something smaller, as in Figure 5. Clearly the topology of the diagram hasn't changed—the way the lines cross each other is the same, so $k(N)$ is the same as for the 90-degree case.

But how about the lengths? A point $(0, \alpha)$ on the vertical axis is connected to point $((1 - \alpha) \cos \theta, (1 - \alpha) \sin \theta)$ on the diagonal axis. The length between them is

$$\sqrt{[(1-\alpha)\cos\theta]^2 + [\alpha-((1-\alpha)\sin\theta)^2]}$$
$$= \sqrt{1-2\alpha+2\alpha^2+2(\alpha-1)\alpha\sin\theta}$$

Summing this up as before over N strings, we get the result shown in Figure 6.

This little figure lends itself to all sorts of nice designs. Figure 7 shows a few examples.

The regular complete graph

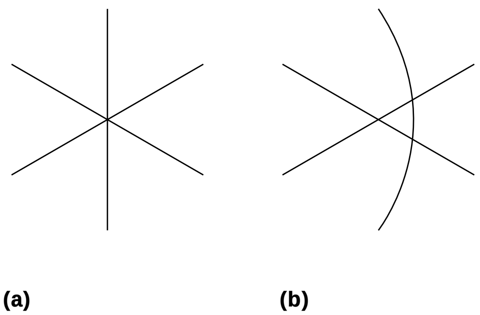
Let's now return to the nails-and-wire construction of Figure 1. I call this a *regular complete graph* because it's a complete graph (that is, a graph where every vertex is connected by an edge to every other vertex) based on a regular polygon. Given the nice solution for the string-art examples in the previous section, I wondered if there would be a nice answer for the number of crossings in regular complete graphs as well. There's a bit of a subtlety in that problem statement which needs refinement. I'll describe the problem, then restate the question.

We'll need to establish a couple of conventions, which I'll do by example. Clearly the smallest polygon that has any internal edges is the square. Figure 8a shows the regular complete graph for $N = 4$. We can see that there's only one crossing, $k = 1$. This shows my first convention: I won't count the lines that come together at a vertex as a crossing.

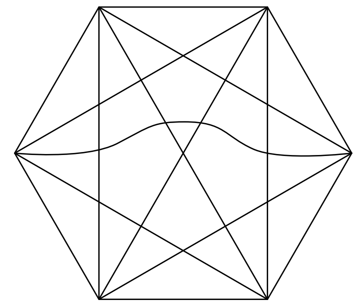
Figure 8b shows the pentagon associated with $N = 5$. It's easy enough to just count the crossings and find that $k = 5$. Figure 8c shows the hexagon associated with $N = 6$. Things are starting to get a little complicated.

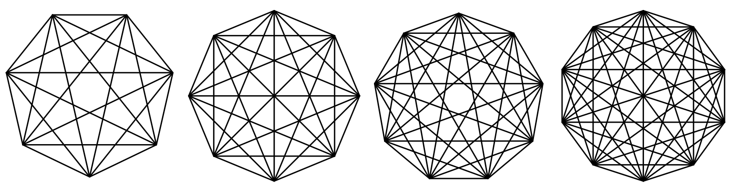
The important thing to note in the hexagon is what's happening in the very center—three "diagonals" cross each other at the same point. I marked this with a red dot. Let's call this a *third-order crossing* (we could also say this crossing has *order three*). Until now, all the crossings have been of order two. How should we count crossings of an order greater than two? This is the problem I alluded to before when we simply wanted to "count the crossings."

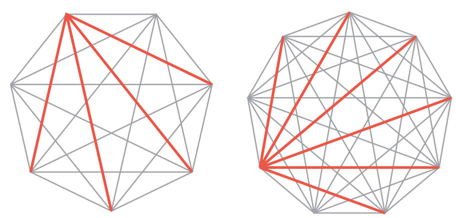
Here's my other convention: We'll "explode" all crossings into a collection of crossings of order two. We'll do this by jiggling the lines around as needed, bending them to one side or the other of the common point so



9 **(a) A third-order crossing. (b) Exploding the third-order crossing into three second-order crossings.**


10 **The exploded complete regular graph for *N* = 6.**


11 **The complete regular graphs for *N* = 7, 8, 9, 10. Notice that the graphs for *N* = 8 and *N* = 10 require exploding to avoid crossings of order more than 2. The cross numbers for these graphs are 35, 70, 126, 210.**


12 **Fans (marked in red) for *N* = 7 and *N* = 9.**

that each crossing has only two lines. Figure 9 shows what happens for a third-order crossing—we get three second-order crossings. It doesn't matter which line you jiggle or how you jiggle it—the result is always the same. Thus I'll count that single point in the center of Figure 8c as the equivalent of three second-order crossings. I found it helpful to sometimes redraw the graphs with wiggly lines rather than straight ones, until only second-order crossings appear, as in Figure 10 for the hexagon. As long as you don't introduce any new crosses in the process of exploding the higher order ones, it doesn't matter exactly how you distort the lines.

From now on, I'll use the term *crosses* to refer only to second-order crossings. So here's our repaired problem statement: How many crosses C are contained in a regular complete graph of N vertices?

If you just count up the crosses in Figure 10 you'll find 15. Figure 11 shows a few more complete graphs. By manually counting them several times, I confirmed that for $N = 4$ through 11 the sequence of cross numbers is $\{1, 5, 15, 35, 70, 126, 210, 330\}$. Obviously, it would be nice to have an expression for the cross number $C(N)$ that doesn't require manual counting.

I'd like to define one more term before we get rolling. Consider any vertex of a regular complete graph of N vertices. There are $N - 1$ other vertices, and since there's a line to each one, $N - 1$ lines radiate from each vertex. Two of those lines are part of the graph's perimeter and are never involved in any crosses. I call the $N - 3$ remaining lines the *fan* associated with that vertex. Figure 12 shows some examples. Every vertex in the graph has a fan of $N - 3$ radiating lines.

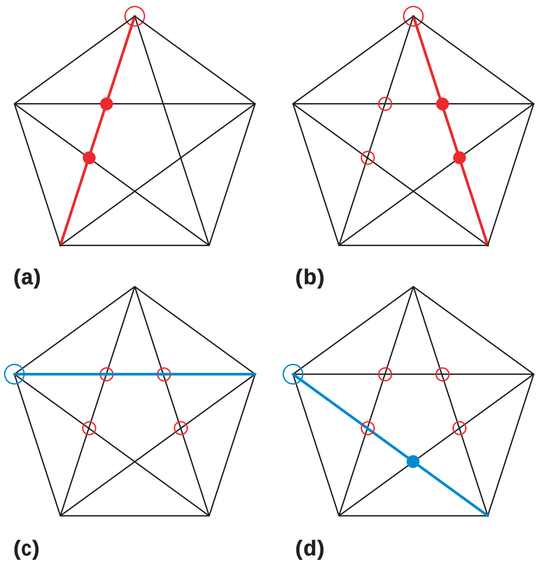
Now that we have our conventions established and terms defined, let's find an expression for $C(N)$. It turns out to be a surprisingly pleasant journey.

One vertex at a time

I'm going to describe a way of counting crosses that leads to an expression for $C(N)$. I'm not going to present a proof, both because I don't have one and because this wouldn't be the right place for the rigor of a proof anyway. But it's a rather straightforward argument. I hope to convince your intuition of the soundness of this approach, and that you could write a proof if you were sufficiently motivated.

We'll take three steps to find the total number of crosses. Here we'll take the first step and see what happens to a graph of size N when we add a vertex to make it of size $N + 1$. Step two comes in the next section, where we'll find a nice way to compute the number of new crosses created by that process. Then step three will wrap things up when we use that result to directly compute the number of crosses in the whole graph.

To begin, consider the simple case of $N = 5$. Let's pick one particular vertex and follow the fan of lines coming out of that vertex. We'll start at one end of the fan and count all the crosses, then move to the adjacent fan, count those, and so on.



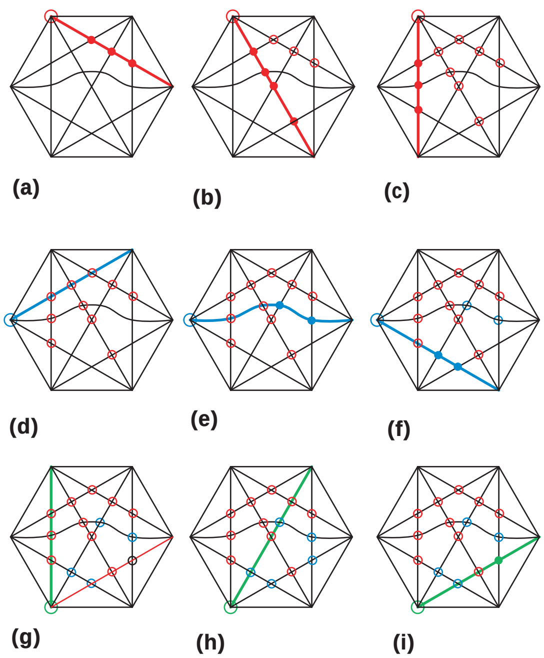
13 **Counting intersections for *N* = 5. (a) Select a vertex (outlined in red) and follow its fan from one end (the line in red). There are two crosses (solid red dots). (b) The second line of the fan. There are two new crosses (solid red dots). (c) The next vertex counterclockwise, shown in blue. There are no new crosses. (d) The next line of the fan, and the last new cross. The sequence for the first vertex is {2, 2} and the second vertex is {1}. The total sequence is {{2, 2},{1}}.**

In Figure 13a I selected one vertex and marked it with a red circle. Our goal at the moment is to count up all the crosses with the fan associated with that vertex. I selected one end of the fan and drew that line in red. Following that line through the graph, we find it contains two crosses. Then in Figure 13b I marked the next line of the fan from that same vertex, and it picks up two more crosses.

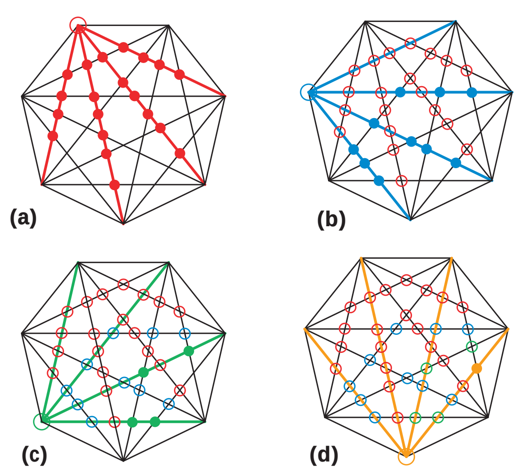
In Figure 13c I repeated the process by moving to the next vertex counterclockwise from the one we just looked at, and marked it in blue. The first line of the fan doesn't pick up any new crosses. In Figure 13d we see that the second line of the fan does create one new cross. None of the remaining vertices contribute any new crosses.

To summarize this, I'm going to create a notation. First, I'll create a sequence of sequences. The main sequence describes the graph, and each subsequence describes one vertex. Within each subsequence, I'll list the number of crosses associated with each fan of the vertex, separated by commas. Any values of zero are ignored. So for $N = 5$, the first vertex we looked at had two lines in its fan, each of which had two crosses, so that vertex is represented by $\{2, 2\}$. The next vertex had zero crosses and then one, so it's simply $\{1\}$. All other vertices were zero. So the complete *cross sequence* for $N = 5$ is $\{\{2, 2\}, \{1\}\}$.

As a thought experiment, you might want to convince yourself that the sequence for $N = 4$ is simply $\{\{1\}\}$.



14 **Finding crosses for *N* = 6. (a),(b),(c) The fan for the first vertex, resulting in sequence {3, 4, 3}. (d),(e),(f) The second fan, resulting in {2, 2}. (g),(h),(i) The third fan with sequence {1}. The total sequence for *N* = 6 is thus {{3, 4, 3}, {2, 2}, {1}}.**



15 **Finding crosses for *N* = 7. (a) The first vertex is {4, 6, 6, 4}. (b) The second vertex is {3, 4, 3}. (c) The third vertex is {2, 2}. (d) The fourth vertex is {1}. The total sequence for *N* = 7 is thus {{4, 6, 6, 4}, {3, 4, 3}, {2, 2}, {1}}.**

Now let's look at $N = 6$, shown in exploded form in Figure 14. The first vertex has a sequence $\{3, 4, 3\}$, shown in Figures 14a, b, and c. The next vertex, marked in blue, has a sequence $\{2, 2\}$, shown in Figures 14d, e, and f. And the vertex after that, in green in Figures 14g, h, and i, is simply $\{1\}$. Thus the sequence for $N = 6$ is $\{\{3, 4, 3\},\{2, 2\},\{1\}\}$.

You may notice something interesting going on with these sequences. If not, take a look at Figure 15, where I marked the results for $N = 7$. The sequence for $N = 7$ is $\{\{4, 6, 6, 4\},\{3, 4, 3\},\{2, 2\},\{1\}\}$.

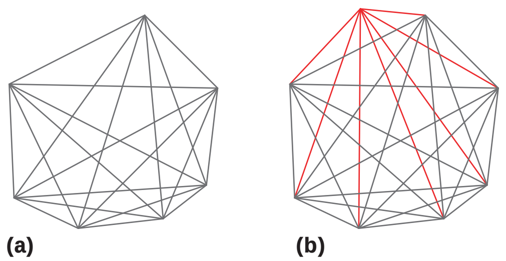
The pattern emerging here is that the sequence for a graph with N vertices consists first of a sequence introduced by that graph, followed by the sequence for the graph with $N - 1$ vertices. Table 2 shows the progression of sequences for a few more values of N, where the pattern shows up pretty clearly.

What's going on here? One way to look at this is to realize that each graph with N vertices can be broken into two pieces: an "old" graph with $N - 1$ vertices and everything left over. The subgraph of $N - 1$ vertices contributes everything but the first subsequence for a graph. How can we characterize what's due to just the "new" part of the graph?

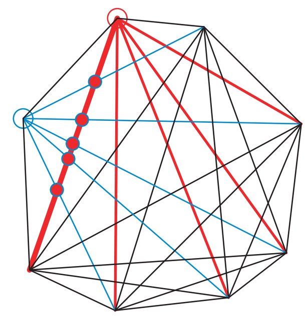
In Figure 16 I took $N = 8$ and split it into two pieces. The $N = 7$ subgraph is in black, while the "new" parts are drawn in red. In Figure 17 I emphasized one of the

Table 2. The sequences for graphs N = 4 through 11.

N	**Sequence for crosses in the regular complete graph of size *N***
4	{{1}}
5	{{2, 2}, {1}}
6	{{3, 4, 3}, {2, 2}, {1}}
7	{{4, 6, 6, 4}, {3, 4, 3}, {2, 2}, {1}}
8	{{5, 8, 9, 8, 5}, {4, 6, 6, 4}, {3, 4, 3}, {2, 2}, {1}}
9	{{6, 10, 12, 12, 10, 6}, {5, 8, 9, 8, 5}, {4, 6, 6, 4}, {3, 4, 3}, {2, 2}, {1}}
10	{{7, 12, 15, 16, 15, 12, 7}, {6, 10, 12, 12, 10, 6}, {5, 8, 9, 8, 5}, {4, 6, 6, 4}, {3, 4, 3}, {2, 2}, {1}}



16 **(a) The graph N = 7. (b) The graph N = 8. The new vertex and edges are in red. The graph is distorted to remove all crossings of degree three and higher.**

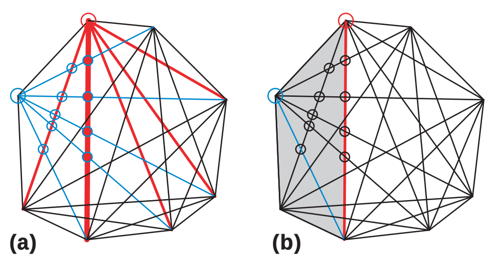


17 **Understanding the counting process. The red vertex crosses each of the lines in the blue fan.**

outermost lines of the fan coming out of the new red vertex with a heavy red line.

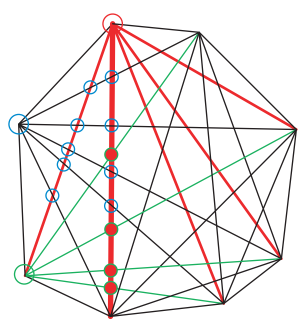
Consider the vertex that's counterclockwise to the red vertex—in Figure 17 I marked this vertex and its fan in blue. It's clear that the heavy red line cuts across the entire blue fan (I marked all six of these crosses). Since in a graph with N vertices each vertex has a fan of $N - 3$ vertices—and obviously the sequences are all symmetrical—we'd expect the first and last numbers in the "new" sequence to be the fan size $(N - 1) - 3$ in the "old" polygon plus the one new edge, or $(N - 1) - 3 + 1 = N - 3$. As a quick sanity check, for $N = 8$, we'd expect the sequence to start and end with 5. It does. A quick scan of Table 2 suggests that this pattern continues as N grows.

As far as this particular line goes, we don't have to bother checking the fans of the other vertices because it's clear that the thick red line can't cross any of their fans. So the sequence for this vertex is so far just {5, …, 5}, and we can move on to the next line in the red fan.



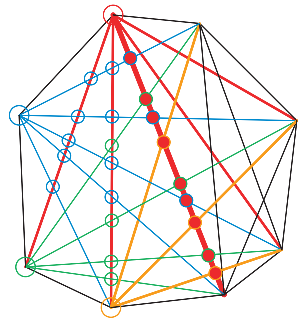
18 **(a) Moving on from Figure 17, the next line in the red fan crosses all but one of the lines in the blue fan. (b) The blue-fan line that is missed and the region of the graph that isn't seen by the heavy red line.**

The thick red line in Figure 18a marks the next step across the red fan. The trick to counting up its crosses is to compare it to the line of Figure 17. The empty circles are the crosses we found in Figure 17. The filled-in circles are the new crosses with the blue fan. The interesting thing to observe is that we've "lost" one cross with respect to the heavy red line of Figure 17, since we're now missing one of the outermost lines of the blue fan. That's because the heavy red line of Figure 18 excludes the little subgraph that contains the "lost" blue line. Figure 18b emphasizes this lost line and shades in the excluded region of the graph. So instead of crosses with all $N - 3$ lines of the blue fan, we only cross $N - 3 - 1 = N - 4$ lines.



19 **Now we add in the crosses formed by the thick red line and the green vertex.**

This line can also intersect the fan of the next vertex going counterclockwise, marked in green in Figure 19. (Note that one of the lines of the green fan overlaps with a line of the red fan. We can dismiss this shared line as a source of a cross.) It's the same story again: The thick red line forms a cross with all $N - 3$ lines of the green fan, except the shared one that's excluded. So the total number of crosses due to the heavy red line are $(N - 4) + (N - 4) = 2N - 8$. For $N = 8$, that's 8 crosses. Table 2 gives the value for other values of N. We've now completely accounted for this red line with 8 crosses, so our sequence for the red vertex is now {5, 8, …, 8, 5}.



20 The nine crosses due to the thick red line and the blue, green, and yellow vertices.

Table 3. Finding the total number of crosses in each fan. Values from $N = 4$ to $N = 11$.

N	Fan Sequence $S(N)$	Sum $S^+(N)$
4	{1}	1^2
5	{2, 2}	2^2
6	{3, 4, 3}	$3^2 + 1^2$
7	{4, 6, 6, 4}	$4^2 + 2^2$
8	{5, 8, 9, 8, 5}	$5^2 + 3^2 + 1^2$
9	{6, 10, 12, 12, 10, 6}	$6^2 + 4^2 + 2^2$
10	{7, 12, 15, 16, 15, 12, 7}	$7^2 + 5^2 + 3^2 + 1^2$
11	{8, 14, 18, 20, 20, 18, 14, 8}	$8^2 + 6^2 + 4^2 + 2^2$

Figure 20 emphasizes the next line of the red fan. The pattern continues. We lose one more cross from both the blue and yellow fans, leaving us with $N - 5$ from each of those. We pick up crosses with the fan of a new vertex, marked in green. But two of its lines have already been taken, leaving us with $N - 5$ crosses from there. So we have a total of $3(N - 5)$ crosses, or 9 for $N = 8$. Again, Table 2 bears out the pattern in general. The remaining lines of the fan are symmetrical.

So the pattern for the "new" fan is $\{\{N - 3\}, \{2(N - 4)\}, \{3(N - 5)\}, \{2(N - 4)\}, \{N - 3\}\} = \{5, 8, 9, 8, 5\}$. I certainly haven't proven anything, but I hope that you'll agree with me that this was a pretty general argument and applies to higher values of N. We can summarize this formally, though since the counting depends on whether N is even or odd, we have to specify two different ways of counting up the elements:

$$c(N, i) = i(N-2-i)$$

$$S(N) = \begin{cases} N \text{ odd: } \{c(N, i)\}, \\ i = (1, 2, \ldots, \lfloor N/2 \rfloor - 1, \\ \quad \lfloor N/2 \rfloor - 1, \ldots, 2, 1) \\ \\ N \text{ even: } \{c(N, i)\}, \\ i = (1, 2, \ldots, (N/2) - 1, N/2, \\ \quad (N/2) - 1, \ldots, 2, 1) \end{cases}$$

If you're unfamiliar with the notation $\lfloor N/2 \rfloor$ in the odd section, it means the "floor" of $N/2$, which in this case simply means its integer component. For example, $\lfloor 7/2 \rfloor = 3$, $\lfloor 8/2 \rfloor = 4$, and $\lfloor 9/2 \rfloor = 4$. This technique pops up all the time in situations where you're dealing with even and odd versions of sequences.

Now we know the sequence that's created when we add a new vertex to a graph. Next, we'll see a nice way to add up all the crosses.

Summing things up

In the last section, we saw that if we knew how many crosses were in a graph with N vertices, we could find the sequence that told us how many new crosses were introduced by adding a vertex to the graph. This suggests a nice way to find the total number of crosses. If we write $S^+(8)$ to mean the sum of all the elements in the sequence created by the first vertex of a graph of size N (so from the above, $S(8) = \{5, 8, 9, 8, 5\}$, so $S^+(8) = 35$), then $C(N)$, the total number of crosses in the graph, is

$$C(N) = \sum_{i=4}^{N} S^+(N)$$

where we can use the formula for $S(N)$ given in the last section.

This would work, and we could stop here, but it's not very satisfying. First of all, finding $S(N)$ itself looks messy. Summing up its elements to get $S^+(N)$ is yet another step. It would be nice to come up with a simpler formula that gives us $S^+(N)$ directly without depending on $S(N)$—then we can use it immediately to find $C(N)$.

Here I'll derive that formula. Again, I won't offer a proof, but I hope I'll show you that the pattern is pretty simple and would be amenable to a proof if you cared to work it through.

Let's look at the various sequences generated by counting crosses along the way. Refer to Table 2 for a summary of the sequences.

To start the process, consider $N = 6$. The hexagon introduces the sequence $S(6) = \{3, 4, 3\}$. Is there a simple way to write the sum $S^+(6) = 3 + 4 + 3$? Well, we could look at this as $3 + 4 + 3 = (3 + 3 + 3) + (0 + 1 + 0) = 3^2 + 1$. Does this do us any good? Consider $S(7) = \{4, 6, 6, 4\}$. Then $S^+(7)$ could be written $(4 + 4 + 4 + 4) + (0 + 2 + 2 + 0)$, or $4^2 + 2^2$. Similarly, given $S(8) = \{5, 8, 9, 8, 5\}$, then $S^+(8) = 5^2 + 3^2 + 1$. Maybe you're starting to see the pattern. I'll write the trailing 1 as 1^2 so that it fits in with the other squares.

This nice pattern results from the construction of the sequences $S(N)$. What's happening is that throughout each even sequence $S(N)$, all $N - 3$ elements are greater than or equal to $N - 3$. When we take those out, we have $N - 5$ elements greater than or equal to $N - 5$. And so on, giving us $(N - 3)^2 + (N - 5)^2 + \ldots 1^2$. The odd sequences work their way down to 2^2 rather than 1^2. Table 3 summarizes this pattern for $N = 4$ to 11 in tabular form. Figure 21 plots the sums.

Well this is nice. We're at our goal for this section, which is a simple formula for $S^+(N)$:

$$S^+(N) = \sum_{i=0}^{\lfloor N/2 \rfloor - 2} (N-3-2i)^2$$

This is really a step forward, since we don't have to mess with $S(N)$ at all—we have a way to get $S^+(N)$ directly. We can now find $C(N)$ by just adding up these elements of $S^+(N)$, as in the summation formula at the start of this section. But it turns out that we can repeat almost the same trick again and get a simple little expression for $C(N)$ itself.

The last step

So far, to find $C(N)$ we sum a series of values. Each of those is the result of a summation. It would be nice to get rid of one of those steps, just as we managed to find $S^+(N)$ without running through all the summations. That's just what we'll do now.

The insight is based on our observation in the last section that each value of $S^+(N)$ is just a sequence of squares. When we add several values of $S^+(N)$, we just get a bigger collection of squares. It turns out that this larger sum has a very nice structure.

Compare $S^+(9) = 6^2 + 4^2 + 2^2$ with $S^+(7) = 4^2 + 2^2$. You're probably way ahead of me and already noticed that $S^+(9) = 6^2 + S^+(7)$. The same thing holds true for the even values. Let's follow the sequence from the start, and the pattern will become pretty clear. Remember that the total number of crosses $C(N)$ for a graph of size N is found from the number of crosses for a graph of size $N - 1$ plus the number of new ones added by the new vertex: $C(N) = S + (N) + C(N-1)$.

We could apply this little recurrance formula directly and be done. But we can also write $C(N)$ without relying on its previous values.

The easy case is $N = 4$. Here, $S(4) = S^+(4) = 1$. The total number of all crosses for $N = 4$ is $C(4) = 1$.

Now $S^+(5) = 2^2$, so $C(5) = S^+(5) + C(4) = 2^2 + 1$. As before, I'm going to write that last term as 1^2, so that everything in sight is squared.

Things start to get interesting with $N = 6$. We know that $S^+(6) = 3^2 + 1^2$. So $C(6) = S^+(6) + C(5) = (3^2 + 1^2) + (2^2 + 1^2)$, or $3^2 + 2^2 + 2(1^2)$.

Moving on to $N = 7$, we find $S^+(7) = 4^2+2^2$. Adding this to $C(6)$, we get $4^2 + 2^2 + 3^2 + 2^2 + 2(1^2) = 4^2 + 3^2 + 2(2^2) + 2(1^2)$.

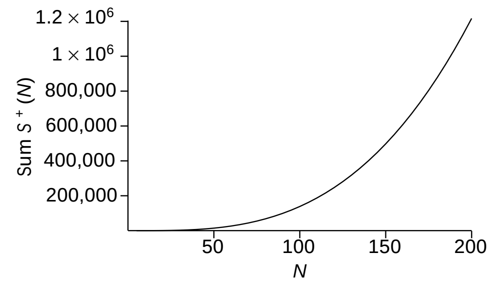
Table 4 summarizes this way of writing $C(N)$. In words, we simply count down from $N - 3$. The first two numbers are squared, the next two are squared and doubled, the next two are squared and tripled, and so on down the line until you reach 1 (which we square as well for consistency).

Table 4 holds the clue to the big payoff. We only need to write this pattern in the form of an equation. That's easily done. And the really nice thing is that there's no need for special cases for even and odd values of N.

Drumroll, please, for the formula for $C(N)$, the total number of crosses in a regular complete graph with N vertices:

$$C(N) = \sum_{i=0}^{N-4} \left(\left\lfloor \frac{i}{2} \right\rfloor + 1 \right) (N-3-i)^2$$

Table 5 shows a tabulation of $C(N)$ from $N = 1$ to 50 in steps of 5. Figure 22 shows a plot of these values. The graph goes up pretty steeply—a regular complete graph of $N = 200{,}000$ has about 6.6×10^{19} crossings.



21 A plot of the function $S^+(N)$ from 1 to 200.

Table 4. A running sum of the rightmost column of Table 3 gives the total cross number $C(N)$.

N	$C(N)$ expressed as sum of squares
4	1^2
5	$2^2 + 1^2$
6	$3^2 + 2^2 + 2(1^2)$
7	$4^2 + 3^2 + 2(2^2) + 2(1^2)$
8	$5^2 + 4^2 + 2(3^2) + 2(2^2) + 3(1^2)$
9	$6^2 + 5^2 + 2(4^2) + 2(3^2) + 3(2^2) + 3(1^2)$
10	$7^2 + 6^2 + 2(5^2) + 2(4^2) + 3(3^2) + 3(2^2) + 4(1^2)$
11	$8^2 + 7^2 + 2(6^2) + 2(5^2) + 3(4^2) + 3(3^2) + 4(2^2) + 4(1^2)$

Table 5. Values of $C(N)$ from 1 to 50 in steps of 5.

N	$C(N)$
5	5
10	210
15	1,365
20	4,845
25	12,650
30	27,405
35	52,360
40	91,390
45	148,995
50	230,300

Wrapping up

There's something really pleasing about discovering graceful patterns. In this column I started with a simple but attractive pattern of lines and moved through a sequence of attractive patterns of numbers. We found the number of crosses in a regular complete graph by a series of relatively simple steps, which were then pretty easy to combine because their patterns were so clear.

Discovering unexpected and graceful patterns in art and mathematics is a joy. And when the math and pictures complement each other, it's particularly delightful to see two different beautiful aspects of the same idea. ■

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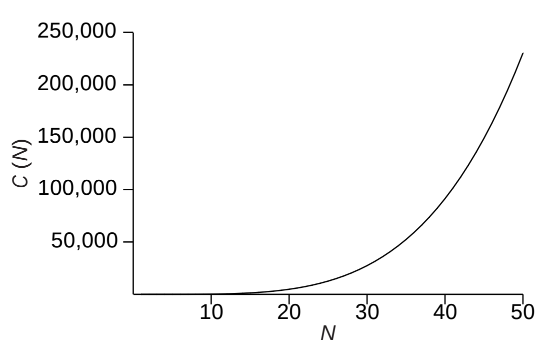

22 A plot of $C(N)$ from $N = 1$ to $N = 50$.