Interactive Pop-up Card Design, Part 1

Andrew Glassner

I love pop-up cards. They're fun to make and receive, and it's a pleasure to watch 3D shapes appear out of nowhere, jump up off the page, and reach for the sky. And as an output medium for 3D computer graphics, they're a perfect match; you get real 3D effects like perspective and parallax, and you don't even need special glasses or other hardware.

I've made and sent out original pop-up cards each time I've moved to a new house over the last few years. You can see some of these cards in Figure 1 (see p. 80).

Anyone who sets out to create their own pop-up cards or books faces two tasks: design and construction. Pop-up design is difficult enough that the really good professionals in the field describe themselves as paper engineers, and they deserve the title.

Creating a great pop-up presents both artistic and technical challenges. Once you have an idea in mind, you need to think about the best way to bring it about. Each pop-up mechanism has its own pros and cons in terms of design time, rigidity, durability, complexity, and construction.

In my experience, coming up with a good idea is just the first step. Typically you want the pieces to unfold and stand on the card's "stage" in just the right way, and this is a delicate matter of just how the pieces are shaped and where they're glued down. If during the design stage you build a piece and it doesn't look quite right, there's nothing to do except cut a new piece with a slightly different shape, glue it in, and see how that looks. This is a time-consuming process of trial and error. When several pieces interact, each change to one piece can have a ripple effect, requiring changes to the others. Each iteration can take a quarter-hour or more, even for a simple card.

Once the design looks right, you have to make sure that the pieces of your card don't jam up while opening and closing. Getting all the pieces to move in the right ways and not get bent or scrunched against each other is a challenging task.

Once you've successfully designed the card, you have to actually make it. This involves cutting out the pieces, decorating them (or affixing decorations to them), gluing them in place, and including any other necessary mechanisms like grommets or string. Making one card is kind of fun, but making 50 quickly becomes tedious.

Each time I've designed and sent out one of my pop-up cards I've wished I had some kind of tool to help me design the cards and some assistants to help construct them. I can't hire the assistants, but I decided to finally go ahead and make the tool in the form of a pop-up design assistant. In this column and the next I'll talk about the issues involved in designing and writing a pop-up design assistant.

My goal in creating my assistant wasn't to create pop-up cards for viewing on the computer. That might be fun, but it seems to miss the point, which is the tactile pleasure in opening the card and the delight of feeling it open. The idea is to make it easier to design wonderful cards, which we can then construct and share in the real world.

In this column I'll talk about how pop-up cards work. (Please see the sidebar—"A Bit of History"—to see how pop-up cards evolved.) It may surprise you to discover that nearly all pop-up cards have only a few basic mechanisms behind them, although they're often combined in unexpected ways. I'll also cover some of the basic geometry behind the most fundamental form of pop up. This will set the stage for writing the program, which I'll discuss next time.

Basic pop-up mechanisms

The best way to learn about pop-up books is to study some great ones: buy great pop-up books and carefully disassemble them, learning from the best paper engineers by studying their constructions.

There are a few good books on designing pop-ups (see the "Further Reading" sidebar). There are easily dozens of techniques—I reviewed my own work and counted at least 3 dozen different kinds of mechanisms. That's a lot. The good news is that most of these techniques are based on just a handful of basic ideas. Think of pop-ups like a guitar. There are only so many ways to pluck and strum the strings and rap on the body of the instrument, yet great guitarists can create a tremendous range of personally expressive and distinctive styles.

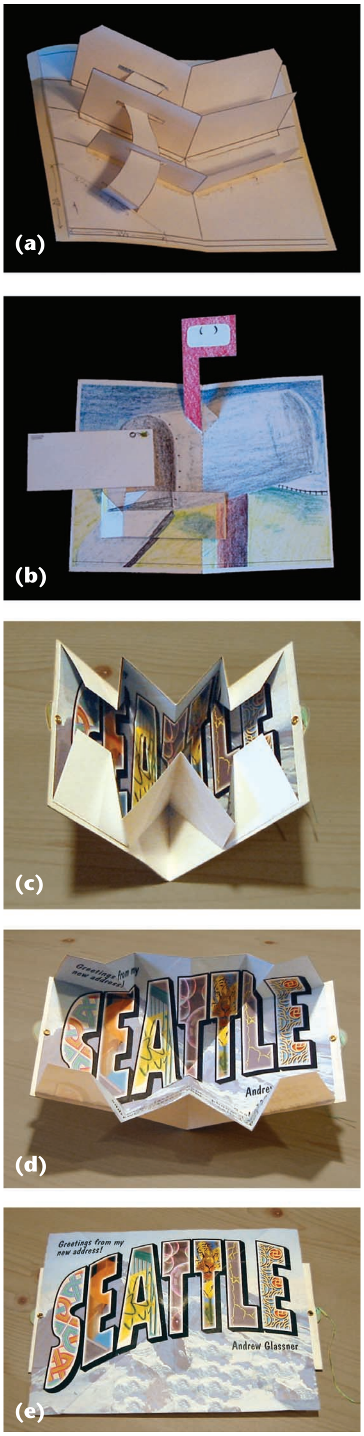
In this column, I'll limit my attention to cards based on stiff sheets of paper, so I'm ruling out bending and curling as deliberate design elements.

It's important to keep the complexity of the card under control. I know from personal experience that the complexity of a card has a huge impact on how long it takes to build. Remember that each cut and fold will have to be repeated for each card. Many cards have multiple pieces, so it's not unusual to spend an hour per card for even the simplest pop-ups. It's easy to design a card that would take an afternoon to build. If you're making only one or two (like a special birthday card), this can be a fun weekend or evening project, and the extra complexity can be fun for you and the receiver alike. If you're making lots of cards, then getting the most out of simple techniques becomes essential.

90-degree mechanisms

The *single-slit mechanism* is part of the class of *90-degree techniques*. That means that they're at their most effective when the card is open to a right angle. When the card is fully closed, of course, there's nothing to see, and when it's fully open, these mechanisms retreat into the plane of the card itself.

In the single-slit method, we fold the card and make a single cut. Typically we also make a single corresponding fold from the cut's edge to the card's central fold. Then we open the card partly, push the cut-and-folded section forward until it snaps into the forward



1 Some of my own pop-up cards announcing recent changes of address. (a) The *blank*, or *mechanical*, for a multilevel card. The three V-folds represent mountains, and the curvy path that passes through them was the road I drove across the country. The road pulls into place by the V-folds' motion. The final version of the card had artwork on all the pieces. (b) Opening this card causes the envelope to tilt out of the mailbox and the red flag to go up. (c) This is a no-cut popup that has just been opened. (d) The popup of Figure 1c as it opens. (e) The final version of Figure 1c, before the placement of the new contact information in the lower right.

A Bit of History

Today we have a great variety of pop-up and moving books to enjoy. This wasn't always the case, of course. The history of pop-up books blends together commerce, innovation, book publishing, personal creativity, and even world events. Movable books weren't originally meant for children. In fact, it wasn't until the 1700s that there was any serious trade in children's literature. Before that, all books were for adults and usually serious in subject matter.

What is perhaps the first movable book seems to predate even traditional printed books. In Majorca, Spain, a Catalan mystic named Ramon Llull (1235–1316) drew a book that represented his mystical philosophy using a set of differently sized revolving disks, or *volvelles*. He divided up the world into several categories. Within each category he identified entities as either superior or inferior. Some of Llull's categories included things and ideas, substances, adjectives and verbs, and knowledge and actions. He divided each disk into sectors (like pie wedges) and assigned one theme to each sector. The disks were then cut out and stacked up, so that you need only turn the wheels to understand nature and thus also predict the future.

Volvelles were particularly used for astrology. A Latin manuscript uses volvelles to describe the motion of the planets over an almost 400-year period, from 1428 to 1808. The first use of volvelles in a printed book was *The Calendarium* by Regiomontanus in 1474. This wasn't really a movable book in today's terms, because you had to cut the disks out and assemble them.

Movable disks were also used for mathematics. In 1551, Johannes Schoner—a Nurenberg professor—published a calculator in movable-disk form in *Opera Mathematica*.

Although the volvelle was popular, as early as the 1300s people were also using flap techniques for mechanical books. These were called *turn-up* or *lift-the-flap* books. They were used in many different fields, but perhaps nowhere as much as in anatomy, because you could simulate a dissection simply by raising successive layers. One of the most famous examples of an anatomical movable book is Andreas Vesalius' *De Humani Corporis Fabrica Librorum Epitome*, printed in Basel in 1543. The book presents the chest, abdomen, and viscera through seven highly detailed, superimposed layers hinged at the neck.

All of these books were handmade. Perhaps the first printed movable book was *Cosmographia Petri Apiani*, an astrological book published in 1564.

In the 1700s the economics of printing changed, and gave birth to a new class of literature: children's books. Most of these books told well-known children's stories and fables that presumably everyone already knew. The value of the books was that children could read them for themselves when their parents weren't there. Movable devices made them even more appealing.

In 1765, the London book publisher Robert Sayer created a series of children's books he called *metamorphoses*. Sayer took a single large sheet and folded it to create four panels, each of which could be opened to reveal a different scene or bit of verse. Several of the books featured a character known as Harlequin from pantomime theater, so the books also came to be known as *harlequinades*, or sometimes just *turn-up* books.

In the 1860s, a London publisher named Dean and Sons became the first to devote itself entirely to what was now called the field of children's *toy books*. Dean and Sons created books based on the popular *accordion*, or *peep-show* style. The idea was that many layers were stacked one behind the other. The child opened the book by pulling the layers apart and setting them up on a table. A ribbon connected the layers—it ran through each one and emerged from the rearmost layer. By pulling on this ribbon, the structures in each layer were pulled out and into position. Then the child peered through a hole in the front cover to view the newly created multiplane diorama. From the 1860s to about 1900, Dean and Sons produced about 50 toy books based on this principle. Dean and Sons also developed a crude technique for dissolving one picture to another using a low-resolution Venetian blind effect (or *jalousie*).

By this time high-quality toy books had become a popular luxury item for the children of rich Europeans and Americans. In 1891, a German publisher named Ernest Nister started designing new mechanical books in his Nuremberg studio and printing them in Bavaria, where costs were low and quality was high. Nister refined the Venetian blind effect and extended it to a circular version that he used in a book called *Magic Windows*. Because he could afford to charge high prices, Nister was able to make high-quality books, and became the best-known publisher of movable children's books by the turn of the century. Another of Nister's innovations was that his books didn't require a ribbon. The illustrations stood up automatically when children turned the pages.

Although Nister was the best-known publisher of movable books for children, a contemporary of his named Lothar Meggendorfer was setting a new standard for complexity and ingenuity. Meggendorfer was a mechanical wizard who created tiny metal rivets out of tightly wound thin copper wire. He embedded these rivets inside double-paned pages, and connected them on the outside to colorful, die-cut figures. Simply by pulling on a single tab, the reader caused the figures to move in elaborate ways in many different directions at once. Some of the actions were even staged to occur in time-delayed sequences as the wire uncoiled from one rivet to the next. Meggendorfer's books were widely praised, as much for their humorous visuals and verses as for their innovation and complexity. Even today, Meggendorfer's works are considered some of the finest movable books ever made. His book *The Circus* has been described as one of the most sought-after books of the 19th century. Between 1878 and 1910 Meggendorfer wrote and designed more than 300 complex, funny, and innovative mechanical books.

World War I put an end to what is now considered the Golden Age of mechanical books.

After the war, the British publisher Louis "The Wizard" Giraud revived movable books. From 1929 to 1949, he produced 16 annual books named the *Daily Express Children's Annual*, as well as several books called the *Bookano Stories*. These were called *dimensional books* because they were mostly about depth and perspective, rather than moving parts. He also called them *living models* because he designed them to be viewed from several different directions—like today's pop-up books—rather than through a pinhole or from just one point.

Giraud's books delivered two other innovations. First, they were the first to lift by themselves when the book was opened 180 degrees. Second, the action sometimes continued even after the book was open. For example, in one particularly clever construction, opening the pages reveals a clown swinging on a trapeze. Even after the book is completely open, the clown continues to swing back and forth. Although uncredited, it appears that Theodore Brown, an inventor who also worked on motion pictures, was the paper engineer who constructed these surprises.

In 1932, the term *pop-up* first appeared. The American publisher Blue Ribbon Publishing of New York created a line of illustrated Disney storybooks created by the Ohio artist Harold Lentz, which they called *pop-up books*.

The economics of children's book publishing and mechanical book construction changed for the better in the 1960s. Julian Wehr created a series of movable books featuring colorful, articulated people that moved in response to pulling a tab. At the same time, Czechoslovakian artist Viotech Kubasta created dozens of popular pop-up books based on fairy tales.

Today, many English-language books are designed in Europe and America, but almost all are printed and constructed in Columbia, Mexico, and Singapore, where the tedious and painstaking cutting and assembly steps are less expensive.

Publisher Waldo Hunt has estimated that from 1850 to 1965 a total of less than 10 million pop-up books were produced in the entire world (http://www.intervisualbooks.com). Today, up to 25 million mechanical books are published annually, with 200 to 300 new titles appearing in English every year. Many chain bookstores now have an entire section devoted to pop-up and movable children's books.

Pop-up and movable books have also become popular for adults again. Publishers are discovering that they're a great way to show complicated spatial relationships, as well as surprise and entertain adult readers.

position, and we've created a pop-up!

Figure 2a (on p. 83) shows a paper model of the most basic single-slit popup: the right-angle single slit with a fold to the crease. Basically all that's happening here is that segment of the card is bending away from the fold rather than along it.

Despite its simplicity, this mechanism contains most of what we need to know about the geometry of pop-up cards. It also has a lot of flexibility, as Figures 2b through 2d show (technically Figure 2e is a *double-slit* design, but the idea is the same).

Before we dig into the geometry, though, we should make sure that it's a reasonable course of action. I investigated two approaches to pop-up geometry: *constraint systems* and *explicit modeling*.

A constraint system is a general-purpose program that finds values for a set of variables, so that those values satisfy a certain list of requirements, or *constraints*. For

Further Reading

There are a few books on paper engineering that should be in the library of anyone who's thinking of getting involved in the field. A terrific listing of all the essential mechanisms—complete with working examples—is in David A. Carter's and James Diaz's *The Elements of Pop-Up* (Simon & Schuster, 1999). Another book offers fewer mechanisms, but gives more detail on each one, including some preprinted pages for you to cut out and fold. This book is *Paper Engineering* by Mark Hiner (Tarquin Publications, 1985). Another great survey of the essentials, with dozens of suggested projects, is in *The Pop-up Book* by Paul Jackson (Henry Holt, 1993).

Everyone has their favorite pop-up books. Like any other kind of book, what one person loves can leave another bewildered. But here are some pop-up books that I believe most people will find entertaining, or at least interesting. These are by no means exhaustive lists, nor do they include all my favorites—that would take pages. Rather, they're just good jumping-off places into the literature.

Some great pop-up books for children include *Robot* by Jan Pienkowski, paper engineering by James Roger Diaz, Tor Lokvig, and Marcin Stajewski (Delacorte Press, 1981); *Haunted House* by Jan Pienkowski, paper engineering by Tor Lokvig (Dutton, 1979); *Alice's Adventures in Wonderland* by Lewis Carroll, illustrated by Jenny Thorne, paper engineering by James Roger Diaz (Delacorte Press, 1980); and *Monster Island* illustrated by Ron Van der Meer, paper engineering by Tor Lokvig and John Strejan (Hefty Publishing, 1981).

Some artists do both the illustrations and paper engineering for their books. Some great examples include *Sam's Pizza* by David Pelham (Dutton, 1996), *The Movable Mother Goose* by Robert Sabuda (Little Simon, 1999), *Bed Bugs: A Pop-up Bedtime Book* by David A. Carter (Little Simon, 1998), and *Chuck Murphy's One to Ten Pop-up Surprises!* by Chuck Murphy (Little Simon, 1995).

A couple of recent pop-ups for adults include *The Human Body* by Jonathan Miller and David Pelham (Intervisual, 2000), and *The Pop-up Book of Phobias* by Gary Greenberg, illustrated by Balvis Rubess, paper engineering by Matthew Reinhart (Rob Weisbach, 1999).

Three library Web sites were invaluable to me in compiling my history of pop-up books. *Moving Tales: Paper Engineering and Children's Pop-Up Books* is a record of the Foyer exhibit in the State Library of Victoria in 1995 (available at http://www.vicnet.net.au/vicnet/book/popups/popup. html). A short but very readable account appears in the Rutgers University Web site *A Concise History of Pop-up and Movable Books* by Ann Montanaro (http://www.libraries.rutgers.edu/rulib/spcol/montanar/p-intro.htm).

The University of North Texas has two great sites that contain a ton of information. But even better, they contain many photographs of mechanical books through history. They show animated versions of books being worked and even provide videos of some books being opened. They appear at *Pop-up and Movable Books: A Tour Through Their History* (http://www.library.unt.edu/rarebooks/exhibits/popup2/default.htm) and *The Great Menagerie: The Wonderful World of Pop-Up and Movable Books, 1911-1996* (http://www.library.unt.edu/rarebooks/exhibits/popup/main.htm).

If you're keen to look more closely at constraint systems, a good place to get started is the book *Solving Geometric Constraint Systems* by Glenn A. Kramer (MIT Press, 1992).

You can find out a lot more about the radical axis and other aspects of circular geometry in Dan Pedoe's *Geometry: A Comprehensive Course* (Dover Publications, 1970).

Some of the material in this column was carried out while I worked at Microsoft Research. That work appears in my technical report, *Interactive Pop-up Card Design* (Microsoft Research Technical Report MSR-TR-98-03, January 1998, http://research.microsoft.com/scripts/pubs/view.asp?TR_ID=MSR-TR-98-03), and is covered by patent 6,311,142.

I'm not the first person to have his hand at bringing together computers and pop ups. You can read about a rather different approach in the article "Mathematical Modelling and Simulation of Pop-Up Books," by Y.T. Lee, B. Tor, and E. L. Soo (*Computers & Graphics*, vol. 20, no. 1, 1996, pp. 21–31). Another article on paper manipulation that is relevant to pop-up techniques is "Bending and Creasing Virtual Paper," by Yannick L. Kergosien, Hironoba Gotoda, and Tosiyasu L. Kunii (*IEEE Computer Graphics and Applications*, vol. 14, no. 1, Jan. 1994, pp. 40-48). A great reference on paper and its geometric properties is "Curvature and Creases: A Primer on Paper," by David A. Huffman (*IEEE Transactions on Computing*, vol. C-25, no. 10, Oct. 1976, pp. 1010-1019).

example, given variables A, B, and C, we might require $A > B$, or $A = B$, or $A + B > C$.

Constraint systems are flexible tools for solving complex problems. But they have three big drawbacks for this application: they are typically large and difficult to write and debug, they are notoriously sensitive to numerical instability, and they can get struck while searching for a solution and end up with no solution at all.

The simpler alternative I followed was to write special-purpose code to explicitly calculate the pop-up cards' geometry. This approach has a few things going for it: the geometry is relatively straightforward, so the code is easy to write and debug, it's fast, and it's stable (it just calculates the proper answer right away).

One downside of writing the explicit geometry in the code is that it limits designers to using mechanisms that have already been prepared. I think this is a reasonable limitation, since there seem to be relatively few mechanisms in general use. If someone cooks up something surprisingly new, then it can be added to the library.

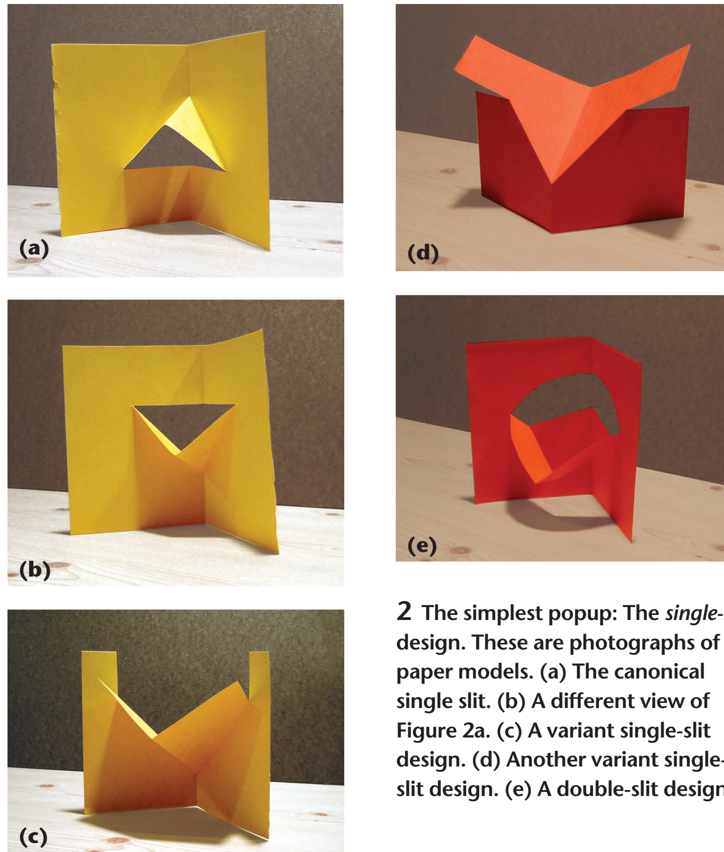
Let's look at the geometry of a single-slit pop up. Figure 3 shows the essential geometry behind all single-slit designs. We begin with a *card*, or *backing plane*. Point A is on the *central fold*, and points C and D lie at equal distances from the fold along a line perpendicular to it. We score the card along lines AD and AC, and cut along CD.

Before we make the cut, the line CD crosses the fold at point E. After the cut, I distinguish point E as that spot on the card where the cut crosses the fold, and point B as that point on the paper at the end of the folded segment. When the card is flat, points B and E are theoretically coincident (in practice, of course, B will be slightly closer to A). As the card folds, point B will move in a direction opposite to that of point E, and that's what makes the pop-up pop.

Let's label the right side of the card as plane π_2 and the left as π_1. I'll call the angle formed between these two planes ω, measured from π_2 to π_1, as in Figure 3. The fold line itself is called L_F. As I fold the card, triangle ABC rises; I'll call this plane π_4. Similarly, triangle ABD is π_3.

To make things easier, I'll assume that the left half of the card (plane π_1) is held flat on the table and the right plane (π_2) is opened. This doesn't limit our generality in any way, but it makes it easier to label the points. Points A, D, and E are all constant in this setup because they lie in the unmoving plane π_1. Points B and C do move. I'll label the position of these points for a given angle ω as B_ω and C_ω, respectively, as in Figure 3. Specifically, B_π is the position of B when the card is fully open, and B_0 is the position of B when the card is fully closed. Our goal is to find the position of B_ω for an arbitrary value of ω.

Since L_F refers to line AE, I'll designate line AB_ω as L_ω, which I also call the *central pop-up crease*. I'll call the two edges AD and AC_ω the *induced creases*, since they appear as a result of the pop-up action. This is a right-angle mechanism, because it's at its best when $\omega = \pi/2$.



2 **The simplest popup: The *single-slit* design. These are photographs of paper models. (a) The canonical single slit. (b) A different view of Figure 2a. (c) A variant single-slit design. (d) Another variant single-slit design. (e) A double-slit design.**

Finding B_ω

When the card is open, point B_π lies in the plane next to point E. As the card unfolds (that is, we lift plane π_2 by rotating it around line L_F), point B_ω rises. In this situation, it's easy to observe that B_ω always travels in a circle with the center at point A and radius AE, in the plane that lies between π_1 and π_2. But a more general solution will prove useful later when we consider more complex types of cards.

To find the location of B_ω for any value of ω, let's start with the things we know. We know the positions of points A, D, and E since they're fixed. We can easily find point C_ω since it's just point C rotated around line L_F by $-(\pi - \omega)$. We also know that because the card is stiff, the distances between points A, B_ω, C_ω, and D are constant.

The key insight is to think of the construction in terms of spheres. Clearly point B_ω always lies on the surface of a sphere with center D and radius $|DE|$, since that distance never changes and the line pivots around point D. Let's call that sphere S_D. Similarly, B_ω lies on sphere S_A with center A and radius AE. Point B_ω also lies on sphere S_C with center C_ω and radius $|DE|$ (since $|C_\omega E| = |DE|$).

Because point B_ω lies on the surface of three different spheres, if we could find all the points of intersection of these three spheres, we would know that B_ω was somewhere in that list.

Three different intersecting spheres that aren't degenerate (that is, they don't have a radius of zero, and none are identical) intersect in exactly two points. Of course, one or more of the spheres could fail to intersect with the others, but in our case we know they do, since we're working from a physical construction. If our three spheres S_A, S_D, and S_C, don't intersect, then our card has come apart!

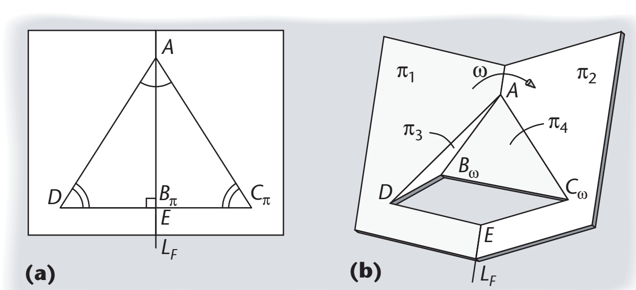
I don't know of a standard solution for the problem of finding the intersection of three spheres, but I cooked up one that is simple, stable, and easy to implement.

To begin with, remember that the implicit formula for a sphere with center C and radius r says that all points P on the sphere equal 0:

$$(P_x - C_x)^2 + (P_y - C_y)^2 + (P_z - C_z)^2 - r^2 = 0$$

So when we plug in B_ω for P into each of the three sphere equations, it will be zero for all of them.

Now imagine a plane through the centers of our three spheres, as Figure 4 (next page) shows. I'll call this π_S. Symmetry tells us that the intersection points of the three spheres lie on a line that's perpendicular to π_S. We mark the intersection point of that line with π_S with a dot.



3 **Basic geometry of the single-slit mechanism.**

The three spheres turn into circles in the plane π_S. When we plug the marked point into those three circle equations, it will have the same value with respect to them all (note that the value won't be zero, since the point doesn't lie on the circles themselves).

Let's pick any two of these two circles and call them *U* and *V*:

$$U(x, y) = (P_x - U_x)^2 + (P_y - U_y)^2 - r_U^2$$
$$V(x, y) = (P_x - V_x)^2 + (P_y - V_y)^2 - r_V^2$$

Now let's look at the structure of all points *P* that have the same value with respect to these two circles. That is, we want to find all points where $U(P) = V(P)$, or equivalently, $U(P) - V(P) = 0$. I'll write this difference of two circles, expand their definitions, and collect like terms:

$$\begin{aligned} 0 &= U(x, y) - V(x, y) \\ &= (x - U_x)^2 + (y - U_y)^2 + U_r^2 \\ &\quad - [(x - V_x)^2 + (y - V_y)^2 + V_r^2] \\ &= x^2 - 2U_x x + U_x^2 + y^2 - 2U_y y + U_y^2 + U_r^2 \\ &\quad - x^2 + 2V_x x - V_x^2 - y^2 + 2V_y y - V_y^2 - V_r^2 \\ &= 2x(V_x - U_x) + 2y(V_y - U_y) + U(0,0) - V(0,0) \\ &= Ax + By + C \end{aligned}$$

Recalling that $Ax + By + C = 0$ is the equation of a line, we've just discovered that all points (x, y) that have the same value with respect to both circles lie on a straight line. This line is called the *radical axis*.

Now let's look back at our three-circle problem, where I'll add in circle *W*. Figure 5 shows these three circles in the general case, where they have different radii and intersect. Circles *U* and *V* meet in two points, which I've labeled P_{UV} and Q_{UV}. Because both of these points have the value 0 with respect to both circle equations, they must lie on the radical axis of those two circles. In other words, to find the radical axis for circles *U* and *V* we need only find their intersection points P_{UV} and Q_{UV}. I'll call this line L_{UV}.

Similarly, I've labeled points P_{UW} and Q_{UW} at the intersections of circles *U* and *W*, and the same thing for circles *V* and *W*. These two pairs of points respectively define the radical axes L_{UW} and L_{VW}.

Now since the circles intersect, L_{UV} and L_{UW} must meet. We'll call that point *M*. Since *M* is on the radical axis between *U* and *V*, $U(M) = V(M)$. And since *M* is on the radical axis between *U* and *W*, $U(M) = W(M)$. Thus $V(M) = W(M)$, which means that *M* also lies on the radical axis L_{VW}.

This little bit of reasoning proves that if three circles are mutually intersecting, then their radical axes intersect at the unique point *M*.

We're halfway home now. Our next step is to locate point *M*, given the three spheres. I do this by creating a plane for each pair of spheres. The plane contains the radical axis and is perpendicular to the plane that joins their centers. So for example, the plane for spheres *U* and *V* contains line L_{UV} and comes out of the page in Figure 5.

To find this plane, take a look at the geometry in Figure 5a. In this figure, we're given the centers of circles C_1 and C_2, their radii r_1 and r_2, and the distance $d = C_1 - C_2$. Of course, this is all the same information as the center, radii, and distances of the spheres. We want to find point *J*.

From triangle PJC_1 we see that $a = r_1 \cos \alpha$. To find $\cos \alpha$ we can use the law of cosines with C_1PC_2 to find

$$\cos \alpha = (d^2 + r_1^2 - r_2^2)/(2r_1 d)$$

so

$$\begin{aligned} a &= r_1 \cos \alpha \\ &= (d^2 + r_1^2 - r_2^2)/(2d) \end{aligned}$$

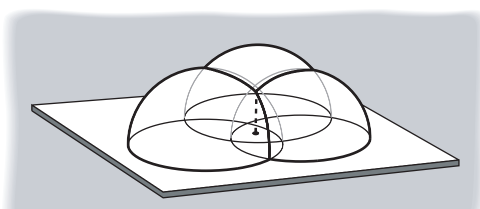
Using these values for *a* and *d*, we can find

$$\begin{aligned} J &= C_1 + (C_2 - C_1)(a/d) \\ &= C_1 + (C_2 - C_1)\,\frac{d^2 + r_1^2 - r_2^2}{2d^2} \end{aligned}$$

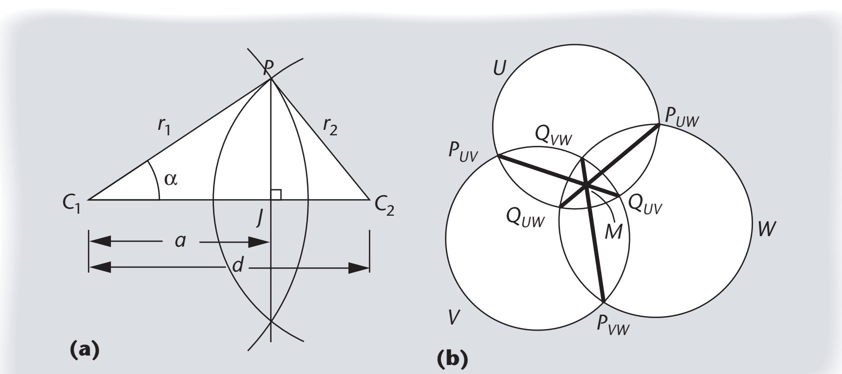
Our plane passes through *J* with a normal parallel to $C_2 - C_1$. Intersecting any two of these planes gives us the dashed line of Figure 4.

With this line in hand, we need only intersect it with any of the spheres to find the two points of intersection, one each above and below the plane π_S. Which point do we want? Refer to Figure 3. We want the point that's on the same side of plane π_S as point C_ω, which we know.

And that point, finally, is B_ω. This may have sounded like a long road,



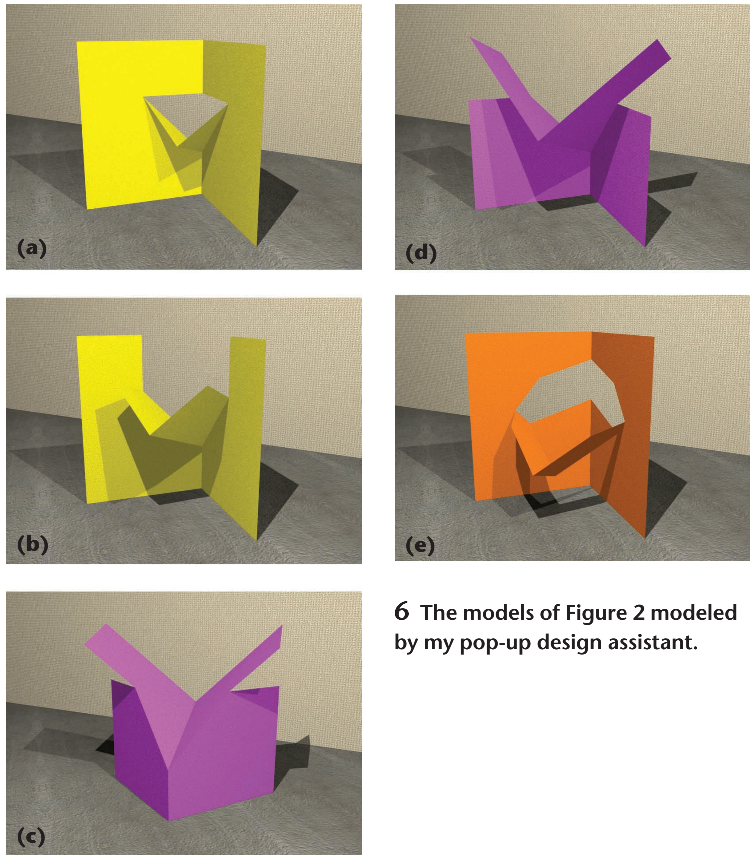
4 Three mutually intersecting spheres. The plane π_S joins their centers. The two intersection points of the three spheres lie on the dashed line that's perpendicular to the plane.



5 The geometry for finding the point of mutual intersection of three spheres. (a) Finding the radical axis through points *P* and *J* for the two circles with centers C_1 and C_2, radii r_1 and r_2, and distance $d = |C_1 - C_2|$. (b) Intersecting the three radical axes.

but most of it was setting up the situation and figuring out what the geometry of the situation looked like. Now that we have the solution, the code itself is pretty short.

If the circles of Figure 5 don't mutually intersect, then the radical axes are parallel and there's no intersection point. But this never holds in our situation. The most extreme case is when the card is fully open or closed, and the circles are tangent—they never fully separate.



6 **The models of Figure 2 modeled by my pop-up design assistant.**

Interactive math

Now that we can find B_ω, we can draw the card for any given value of the opening fold angle ω. Just find the point C_ω, then intersect the spheres to find B_ω, and draw the polygons.

Suppose that we don't like the way the card looks. Then we can simply grab B_ω and move it around interactively. There's only one geometric limit the system needs to enforce on the user: point B_ω must lie on the plane that's halfway between π_1 and π_2 at any stage of folding. We don't even require that the corresponding point E be on the card itself. There's no reason not to let the designer create a card that pokes up from the bottom, or down from the top, as Figure 2 shows.

Finding B_ω is the heart of my pop-up design assistant. Figure 6 shows the results of using my program to recreate the paper models of Figure 2. The routine is short, based on the geometry we discussed. I'll give details on programming it in the next issue.

Asymmetric slits

An important variant of the single slit is the *asymmetric slit*. Here the fold doesn't follow the crease of the backing card—it's at an angle to it. This gives the designer more freedom to create slanted and forced-perspective effects.

Figure 7 shows the essential geometry. Figure 7a is the open card and Figure 7b shows it in the closed position. In Figure 7a, the central pop-up crease AB_π forms an angle β to the support crease AE. Although in action the card looks generally like Figure 2b, the central pop-up crease is rotated, creating an asymmetrical pair of triangles on each side. In Figure 7a, we're free to choose A, D, and C_π. We want to find B_π that lets the card fold flat. In terms of angles, we have ψ, γ, and δ and wish to find α.

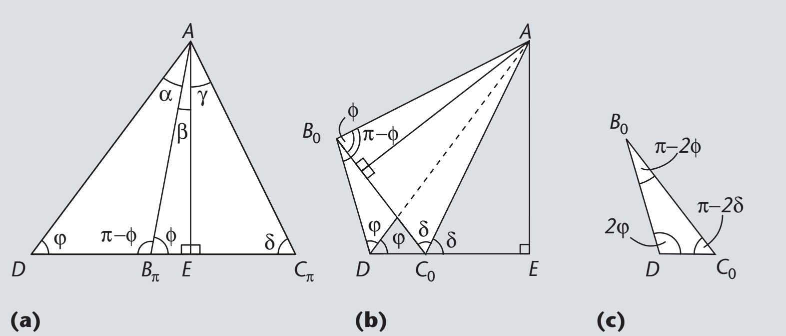
In Figure 7, we can see that as the card folds, point B_π comes up out of the plane and eventually comes down to rest at B_0. This causes triangle ΔADB_π to become reflected, since B_ω pulls it around AD. The motion of B_ω pulls along triangle $\Delta AB_\pi C_\pi$, and comes to rest at ΔAB_0C_0 in an orientation equal to a rotation of γ around A. Because EC_π is perpendicular to the folding axis AE, point C_π moves to C_0 along line DE. This means that triangle ΔAC_0E is similar to triangle $\Delta AC_\pi E$.

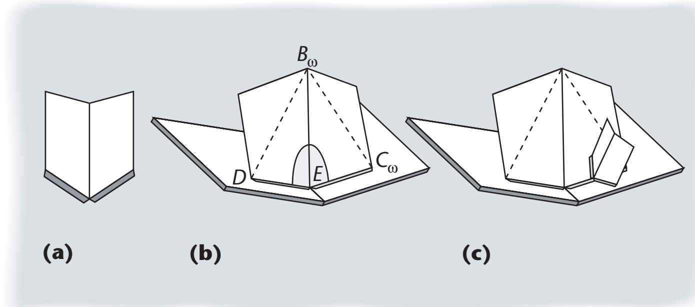
To find α, we begin with ΔB_0DC_0 in Figure 7b and 7c, giving $2\psi + (\pi - 2\delta) + (\pi - 2\phi) = \pi$, or $\phi = \psi - \delta + (\pi/2)$.

From ΔADB_π in Figure 5a, write $\alpha + \psi + \pi - \phi = \pi$. With the value we found for ϕ, this becomes $\alpha = (\pi/2) - \delta$. From ΔAEC_π we then find that $\delta = (\pi/2) - \gamma$. By combining these last two results, we find our goal: $\alpha = (\pi/2) - ((\pi/2) - \gamma) = \gamma$.

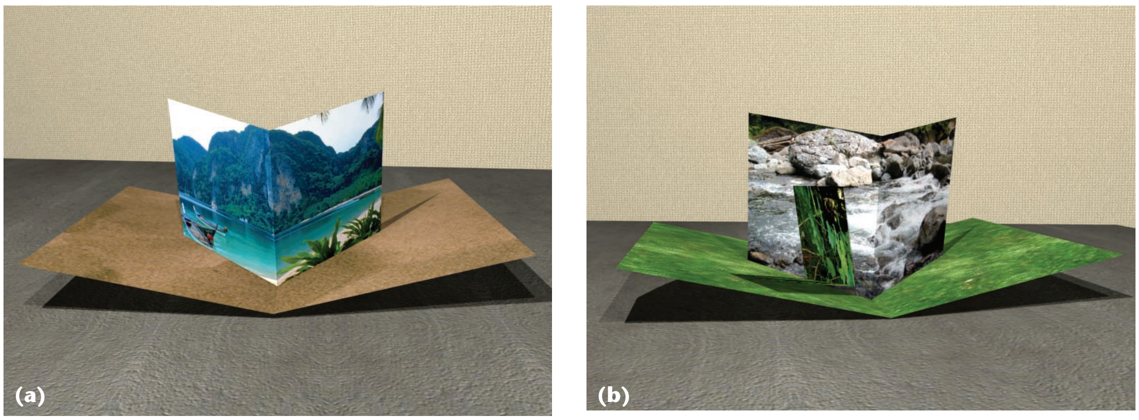
This was a long road that ends with a simple conclu-

7 **Geometry of the asymmetric single-slit.**





8 **V-fold mechanisms. (a) The basic V-fold. (b) How it sits on the card. (c) A second-generation V-fold.**



9 **Computer-rendered V-folds. (a) A single fold sitting atop a backing card. (b) A second generation, or cascaded, V-fold in the process of opening.**

sion: to construct an asymmetric slit pop up that folds flat, place B_π in Figure 7a so that $\alpha = \gamma$.

V-fold mechanisms

The *V-fold* mechanism creates a pair of free-standing slanted planes when the card opens, as Figure 1a shows. The V-fold is one of the hardest pieces to design using paper and scissors, since you indirectly control how much the plane leans back by changing the angle at the base of the piece when it's cut out. This angle is the V at the bottom of Figure 8a.

Because a V-fold is a separate piece attached to the card backing, it can rise out of the card plane when the card is fully open, unlike the single slit. Thus the geometry of the V-fold is based on the single slit, but allows more flexibility in its design. Though B_ω still locates the central crease, there may be no paper at that point in space. For example, the apex of the fold (point E in Figure 8) need not be included; the shaded tunnel region in Figure 8b can be cut out of the card. Figure 8 also shows the small flaps scored, bent back, and then glued to the support planes.

Since V-folds don't cut into the page, they may be placed on any crease, which we then treat just like the card's crease for that mechanism. Figure 8c shows a cascaded pair of V-folds. The larger one uses the card fold as its support crease, and creates EC_ω as one of its side pop-up creases. The smaller V-fold uses EC_ω as its support crease. So opening the card pops up the big V-fold, which then drives the smaller one to pop up as well.

The tabs of a V-fold must be carefully glued in the right places or the card may not open or close fully.

Depending on the placement of the V-fold on the support planes, we can design it to fold either toward or away from the reader. When the planes of the V-fold become parallel to the support planes, all the folding lines become parallel to one another. This configuration is sometimes called a *floating layer*.

Figure 9a shows a computer-rendered V-fold. In Figure 9b I show a second-generation V-fold raising from the crease between the first V-fold and the card. Higher generations of V-folds work just like their parents, although they require a bit of care in programming to keep track of all the points during the opening and closing of the card.

Next time

This completes our introduction to the basic history and geometry of pop-up cards. Next time I'll talk about other mechanisms and some advanced topics, and also describe some of the features and programming of my pop-up design assistant.

Acknowledgment

I created all of the models in the computer-generated images with my program, and then rendered them in Discreet's 3ds max 4.

Readers may contact Andrew Glassner by email at andrew_glassner@yahoo.com.