Origami Platonic Solids

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There's something very beautiful and elegant about the simple 3D shapes known as Platonic solids. Just looking at them can tantalize and entertain the intellect. When you hold one in your hand, fingertips can join intellect to enrich your understanding of the graceful underlying structure—our bodies know things that our minds can't even name. By actually building these structures, the mind and body can work together to strengthen our intuition about them.

A model to have and hold

The traditional way to build polyhedra from paper is to cut out a flattened version and then to fold it and glue it together. But an entirely new way of building polyhedral models adapts the paper-folding art of origami. Origami polyhedra are assembled without any cutting or gluing. You just fold paper and assemble the pieces by tucking flaps into pockets.

You've probably seen origami creations, and you may even have built a few—most kids have built a hat or "fortune teller" at some point by folding paper. Origami is an ancient Japanese art ("ori" = folded, "kami" = paper). There have been different definitions of just what can be considered origami. Generally, the differences revolve around what else you can do besides folding the paper. In this column, I'll stick with a pretty simple and popular description of origami: You start with a single sheet of paper, and you're only allowed to fold it. The only tools allowed are your own fingers—no pencils, measuring sticks, compasses, and so on. (You can buy origami paper in most craft and art stores.)

There is one variation. We'll create most of our models by assembling several different pieces together. The assembly must not require tape or glue, though for some of the flimsier models you may want to apply a few dabs of glue here and there so that the model doesn't completely unravel if you drop it. (I speak from experience!)

Unit origami

Frankly, I've always found traditional origami rather difficult. The notation is very simple, but I often get lost while following the instructions and just can't figure out where I went wrong. My success-to-failure ratio recently went way up with the help of two new books. The first one talks about how to make traditional models, such as animals. Working from *The Complete Book of Origami* (see the sidebar), I've actually completed every model I've attempted—a bit of a feat for me.

The other big news was Tomoko Fusè's book, *Unit Origami*. The basic idea behind unit origami is to build many instances of a single piece, or unit, and assemble them into a more complicated model. Unit origami is mostly directed toward building geometric polyhedra such as the Platonic solids, rather than organic creatures such as squids or dinosaurs.

The main trick behind unit origami is to build a unit that combines what I call pockets, flaps, and locktabs. A *pocket* is like a kangaroo's pouch: an open burrow that will hold a flap from another unit. A *flap* is a bit of paper that sticks out from the unit, generally shaped the same as the pocket it's intended to mate with. When the flap slides into the pocket, the two pieces are joined together. Sometimes there's a little extra bit that hangs off of a flap, which I call a *locktab*. It generally wraps around a fold in the model to keep the flap firmly in its pocket.

There are generally three types of folded units in unit origami:

- A *face unit* is a piece of paper folded to represent a single face. For a cube, you'd make six square faces, each one from its own piece of paper.
- An *edge unit* spans an edge and includes at least some of the face on each side. For the cube, we'd have 12 edge units, each one probably with a triangle on each side of the edge.

Further Reading

For a basic introduction to origami, I heartily suggest *The Complete Book of Origami* by Robert J. Lang (Dover Publications). If you've conquered that book, some good places to move on to include *Folding the Universe* by Peter Engel (Vintage Books), *Origami* by Robert Harbin (Harper & Row), *Animal Origami for the Enthusiast* by John Montroll (Dover Publications), and *Origami for the Connoisseur* by Kasahara and Takahama (Japan Publications).

The basics of unit origami were first put forth in *Unit Origami* by Tomoko Fusè (Japan Publications). A little light on discussion but presenting a wide selection of units is *3-D Geometric Origami* by Rona Gurkewitz and Bennett Arnstein (Dover Publications). *Origami Boxes* by Tomoko Fusè (Japan Publications) applies unit origami to box-making.

To get involved in the very active on-line world of origami, a good place to start is Joseph Wu's origami page, located at http://www.cs.ubc.ca/spider/jwu/origami.html.

- A *vertex unit* contains a corner of the model and generally includes some edge and some face parts. For the cube, we'd have eight vertex units.

Some models can be built in all three ways. Generally, the trade-offs are stability, complexity, and time. For our hypothetical cube, we would need six, eight, or 12 units respectively for face, edge, and vertex-based assemblies. Generally, it takes longer to build 12 units than six, but assembling a smaller number of units is sometimes tougher because the pieces have to fit together more precisely. Also generally, models built with fewer units are flimsier than their more complex counterparts.

Rules of the game

To get the ball rolling, I'll first cover the little bit of traditional notation we'll need, summarized in Figure 1. Developed by Akira Yoshizawa, this notation is now very standardized; almost all books on origami use it. The symbols in Figure 1 represent about half the total available—it's a very simple and spare system.

Figure 1 (top) shows the instruction and results for a piece of paper with a valley fold and a mountain fold. The arrows show where to make the fold. The dashed line represents a *valley fold*; it forms a valley from your point of view. A *mountain fold* is the crest of a mountain, represented by alternating dashes and dots. Solid lines refer either to edges of the paper or to creases you've already made in it. Gray shading indicates the colored side of the paper.

If an arrow has two arcs, as in Figure 1 (bottom left), you should fold and then unfold the paper, rather than leaving it folded. The symbol to flip the paper over is an arrow with a built-in loop, as in Figure 1 (bottom right).

Work on a hard, flat surface. When you make a fold, get it roughly into position with your fingers, and then once everything is in place, smush it down. This way you can still maneuver a little bit to align the pieces. When the paper seems set up correctly, go over the fold again with your fingernail, making a sharp crease. You'll find the sharper your creases, the better your models will look and the easier they'll go together. It's very important, particularly in unit origami, that your folds be accurate. A little error in alignment early on will magnify on each subsequent fold.

Sometimes models are very fragile during assembly. I use artist's removable tape (made of the same adhesive on the back of those yellow sticky notes) to hold the pieces together while I'm building. It comes off without a trace once the model is assembled.

The Platonic solids

Entire books have been written about the Platonic solids. I couldn't even begin to skim the surface here. Jim Blinn gave instructions on how to find the coordinates of these solids in his column of November 1987 (*IEEE CG&A*, Vol. 7, No. 11, pp. 62-66). More information can be found in almost any book on 3D geometry.

In this column, I will illustrate the construction of each Platonic solid with a different origami unit, so we'll see five different units as we build the five different solids.

The tetrahedron

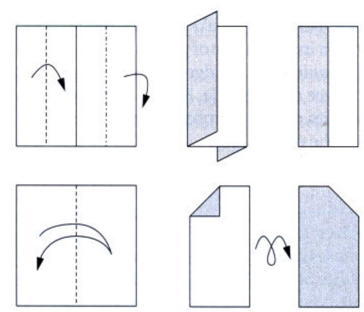
The simplest Platonic solid is the tetrahedron. It has four faces, each an equilateral triangle. Each of the four vertices joins three triangles, which meet along six edges.

We will build this model from rectangular pieces with sides in the ratio 2:1. At first, this might seem an impossible task: How could we start with a rectangular piece of paper and make an equilateral triangle without using a compass, ruler, protractor, pencil, or anything else? Aren't there all sorts of irrational numbers running around inside an equilateral triangle?

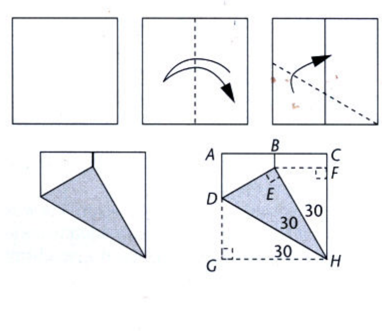
True enough, but this is a special case. Remember the 30-60-90 triangle? If the hypotenuse has length 1, then length of the edge opposite the 30-degree angle is 1/2. This observation is central to folding a tetrahedron.

Let's first see this trick in isolation.

Figure 2 shows a square piece of paper, one unit on a side. We begin by making a valley fold up the middle—just fold the left half over the right, crease, and open it up again. This gives us two vertical rectangles, each of height 1 and width 1/2. Now we fold the bottom-left corner up until it reaches the centerline. The result appears in the bottom left of Figure 2; the geometry of the situation appears in the bottom right. Consider triangle ΔEFH. The original square's edge GH was of length 1, and that edge is the hypotenuse EH of this triangle. Because the hypotenuse is 1 and the far side EF is 1/2, then the angle $\angle EHF$ at the bottom must be 30 degrees; and because angle $\angle CHG$ is 90 degrees, angle $\angle EHG$ is 60 degrees. Now consider the folded triangle ΔEHD. The gap it left behind, triangle ΔDHG, is exactly the same shape. So the angle $\angle EHG$ is bisected, meaning angle



1 Top: Dashed line indicates a valley fold; dash-dot line, a mountain fold. Bottom: An arrow with two arcs means fold to crease, then unfold. An arrow with a loop means turn the model over.



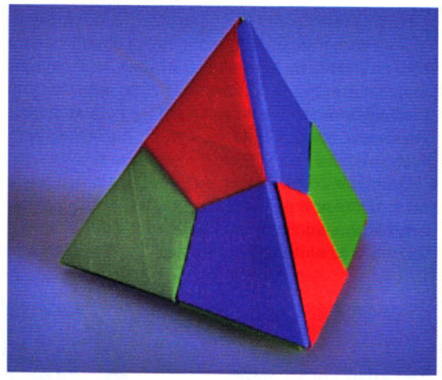
2 Folding a 30-degree angle. Bottom right: The geometry of the fold.

∠EHD is 30 degrees. We have trisected a right angle! Now we can make units with angles that are any multiple of 30 degrees.

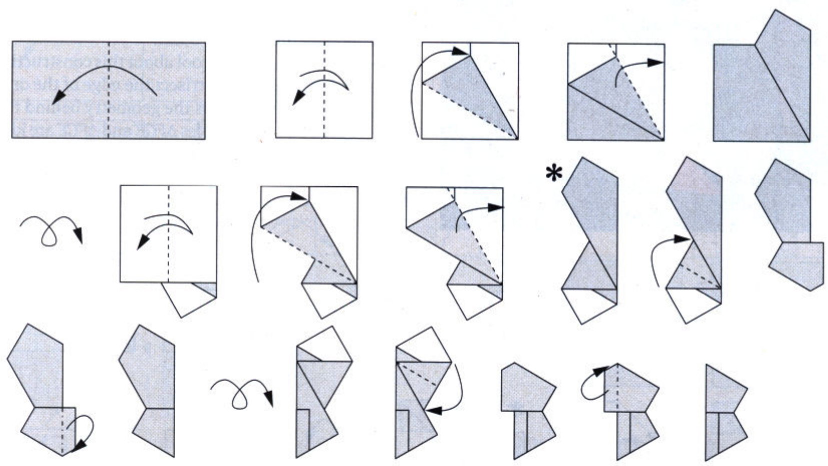
Let's return to the tetrahedron, shown in Figure 3. Fold the unit following the instructions in Figure 4. When you're done, unfold these folds until you reach the starred picture in Figure 4 and then unfold the top half. The piece, seen from the top and bottom, should look like Figure 5a. Assemble the pieces as shown in Figure 5b: Slip the flap of one piece into the pocket of the other, and let the locktab slide down into the bottom of the pocket.

This is the *edge equilateral triangle unit*. The open flap along the diagonal of the piece turns into an edge of the model. Because the tetrahedron has six edges, you'll need to assemble six of these pieces to assemble the model. It can be a little tricky to see how the pieces fit at first, so give yourself a quiet space and some time to play with it. It may feel like a puzzle. In fact, the tetrahedron is one of the hardest models to build, but not because it's complicated. Rather, the folding to get the pieces to lock together seems implausible until after you've done it.

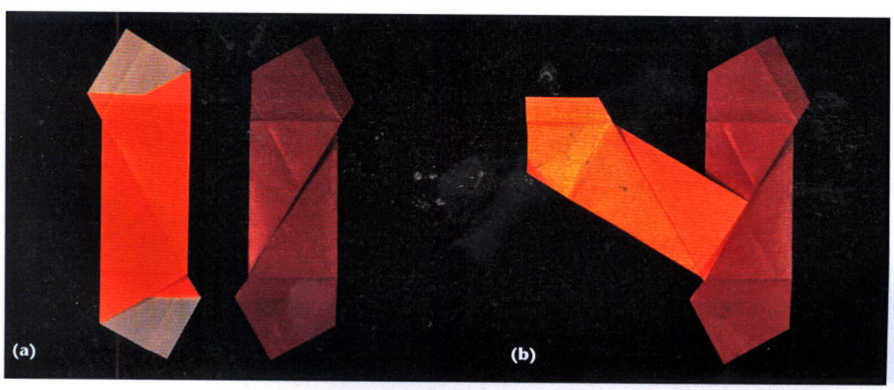
This assembly takes some patience, but you'll eventually get the pieces together as in Figure 3. Then you'll wonder why you thought it was hard! When you're done, every face, vertex, and edge will look like every other (except for color, if you use different-colored papers).



3 Tetrahedron built from edge equilateral triangle units, made from two pieces each of red, green, and blue papers.



4 Folding the edge equilateral triangle unit. When you've completed the folding, unfold back to the step marked with a star.



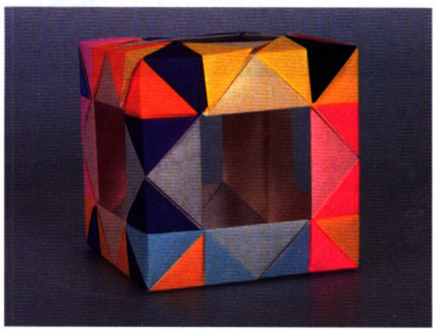
5 (a) The edge equilateral triangle unit. (b) Assembling two units.

The cube

Probably the next simplest Platonic solid, the cube has six faces, twelve edges, and eight vertices. Just as we created the tetrahedron from six edge pieces, we'll create 12 struts for the cube, corresponding to its edges. Figure 6 shows the model, based on what I call the *harlequin strut unit*.

Folding instructions for the strut are shown in Figure 7. Start with a square piece of paper, colored side up. The result is a little right-angled strut with flaps at right angles at the top and bottom, and two pockets along the strut. Figure 8a shows two views of the harlequin strut. I've flattened them for the photograph, but in the assembly, you'll need to fold them into right-angled bars.

To put the struts together, slide the flaps into the pockets, with the new pieces at right angles, as in Figure 8b. Because three units meet at each vertex, you'll need to slide another unit with the first two, as in Figure 8c. The entire cube is assembled by putting together 12 units. This model is particularly floppy as it's going together, so you might want to use artist's adhesive tape to hold the pieces in place during assembly. Once built, though, you can take off the tape and the model will hold together quite well.



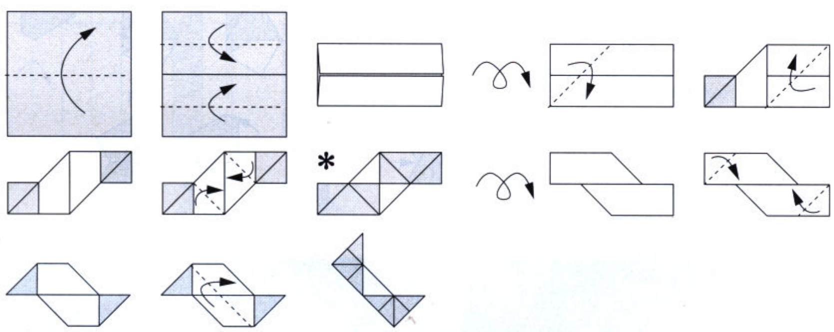
6 Cube built from harlequin strut units.

The octahedron

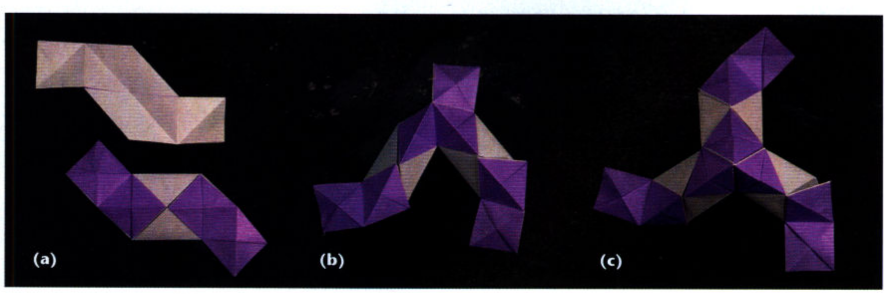
The octahedron has eight faces, each an equilateral triangle. We can build it from 12 of the same edge equilateral triangle units that we used for the tetrahedron. An alternative construction uses only four units, each covering two faces. I call these *double-face equilateral triangle units*. As the name implies, each unit correlates to two faces of the finished solid. Figure 9 shows the octahedron assembled with these units. Figure 10 gives folding instructions, which result in four equilateral triangles in a strip, with locktabs at the top and bottom.

Figure 11a shows the pieces from the top and bottom. To assemble the pieces, line them up as in Figure 11b and slip one flap into one pocket. Then get the other flap into the other pocket, as in Figure 11c. These two pieces now create a square-based pyramid, which forms the top half of the octahedron. It's flimsier than the edge-equilateral-triangle version, but it only takes half the paper and half the effort to build.

There's something very cool about this construction. In the opening section, we trisect the edge of the original square! Figure 9b shows the geometry behind this. As we've seen before, triangles ΔFGE and ΔFCE are identical 30-60-90 triangles. Since the original edge FG has length 1, both CE and EG have length $1/\sqrt{3}$, and the shared hypotenuse EF has length $2/\sqrt{3}$. When we fold



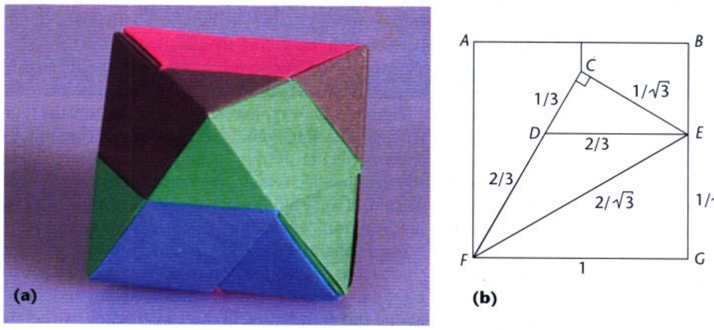
7 Folding the harlequin strut unit. Fold as shown, then unfold back to the step marked with a star.



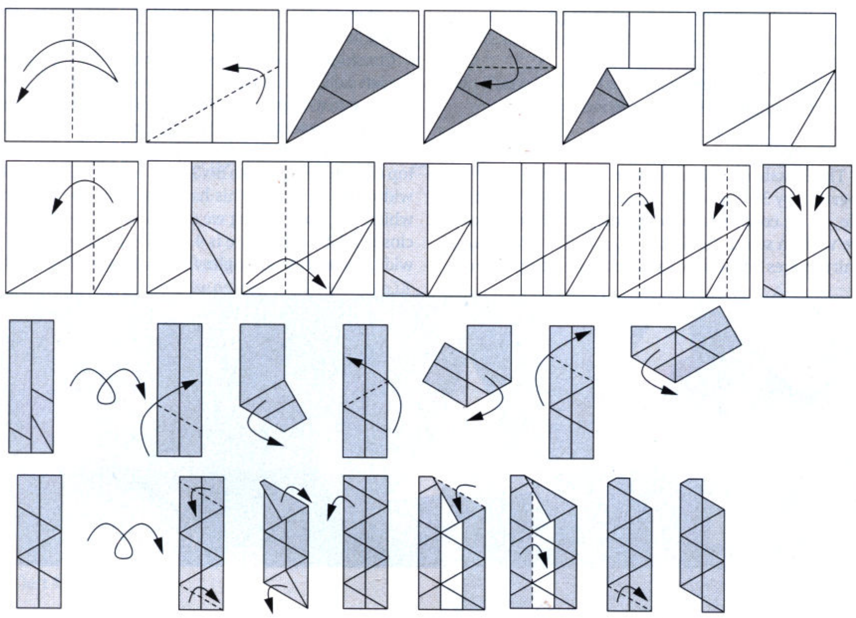
8 (a) Harlequin strut unit. (b) Assembling two units. (c) Adding the third unit makes a single vertex of the cube.

down the point at C to bisect angle $\angle CEF$, we create two new triangles. Triangle CED is another 30-60-90 triangle. Working from the known edge CE (now opposite the 60-degree angle $\angle CDE$), the new hypotenuse DE has length 2/3 and thus the new short side CD has length 1/3. Because CF has length 1, the remaining piece DF must have length 2/3, just like DE. CF is one of the original edges of the square; when we unfold the square, the crease at D marks one-third the square's edge.

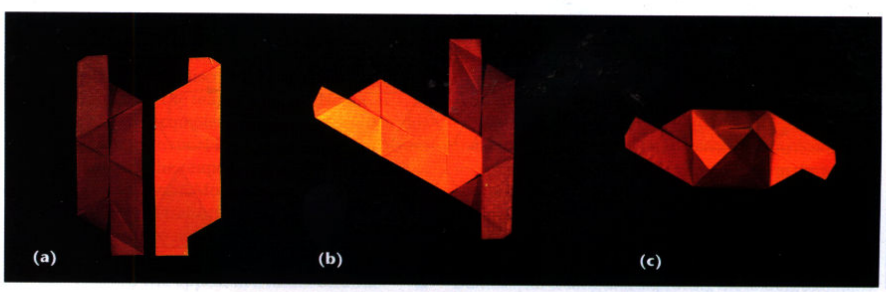
A figure that can be dissected into smaller copies of itself, like this 30-60-90 triangle, is called a *reptile*.



9 **(a) Octahedron built from double-face equilateral triangle units. (b) Geometry of the double-face equilateral triangle unit.**



10 **Folding the double-face equilateral triangle unit. The last fold hides the corner under the right flap.**



11 **(a) Double-face equilateral triangle unit. (b) Assembling two units. (c) Completing the assembly.**

The dodecahedron

The dodecahedron has 12 pentagonal faces, three of which meet at each of the 20 vertices; there are 30 edges. We'll adopt a framework design for it, as we did for the cube. Figure 12 shows the model we will build, using a piece I call the *triangular vertex unit*. We start with an equilateral triangle folded from a square piece of origami paper. Use the opening steps of the tetrahedron construction in Figure 4 to create a 60-degree angle at each end of an edge of the square. The triangle will be formed by the points of the original square at both ends of this edge and by the point where the two creases intersect along the midline.

Purists will want to fold these two pieces in and under to make the triangle. I admit that I use scissors to cut along these folds and make an equilateral triangle-shaped piece of paper. It keeps the resulting unit from being thicker in some places than others.

Figure 13 shows how to fold the triangular vertex unit. Figure 12b shows the pieces from the top and bottom. To combine two units, slide the arm coming out of one into the pocket of another, as in Figure 12c. It doesn't matter which piece goes into which, since the pieces look the same and end up the same thickness when they're assembled.

Since the dodecahedron has 20 vertices, you'll need 20 of these modules to make the model.

The icosahedron

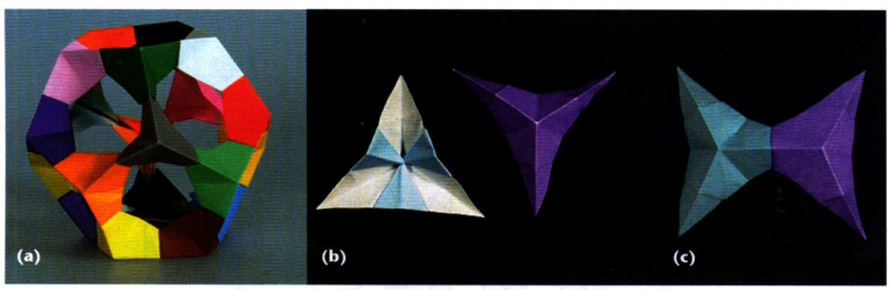
The icosahedron has 20 faces and 12 vertices, held together by 30 edges. Each face is an equilateral triangle, so we could use either of the two triangle forms we've seen so far. Using the edge equilateral triangle unit requires 30 pieces—a whole lot of folding. The double-face equilateral triangle unit is much easier, requiring only 10 pieces to cover the 20 faces.

Just for variety, I'll present an alternative double-face unit, which I call the *four-triangle face unit*. It has a property that crystallographers describe as *enantiomorphism*: There's both a left-handed and a right-handed version (anything with this property, such as a glove, is called an *enantiomorph*). This piece also uses both the front and back of the paper, which tends to make it a little weaker than the other units, so the model is a little floppier during assembly and a little more fragile once built.

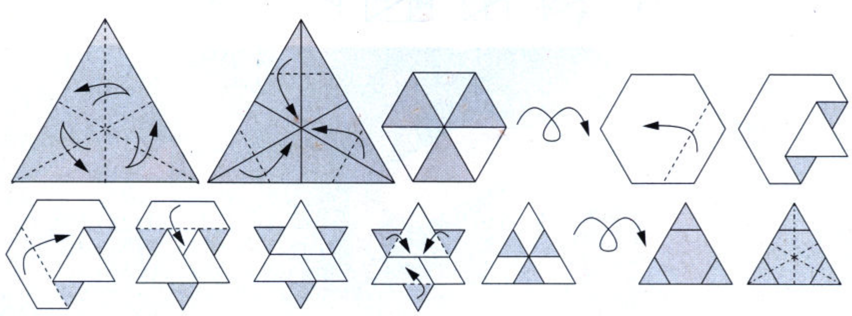
Figure 14a shows an icosahedron built from the four-triangle face unit. Folding instructions for this unit are shown in Figure 15 (the second line shows how to make the right-handed version; the third line, the left-handed version). The resulting unit is seen from above and below in Figure 14b. The units go together as in Figure 14c.

Recall that the previous stack of four triangles—in the double-face equilateral triangle unit—had a little bit of paper left over that was absorbed into the locktab at one end of the stack. In this construction, we get rid of that extra bit of paper by doing all of that initial folding of ever-smaller strips at the top and bottom. This works because of a nice numerical approximation.

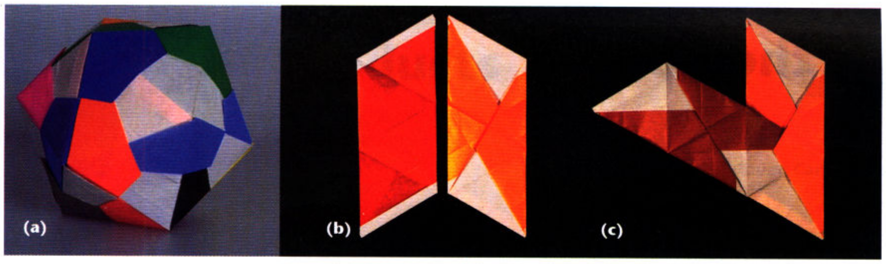
Consider Figure 16. I've built a 30-60-90 triangle on the left side, and then built a couple of equilateral triangles next to it. Suppose that the left edge is 4 units across. Then the hypotenuse of the triangle is $8/\sqrt{3}$, and the short leg on the left side is $4/\sqrt{3}$. So the upper right part of the top edge also has length $8/\sqrt{3}$, for a total of $12/\sqrt{3}$ for the width of the figure. This has a value of about 6.928, which for paper-folding we can think of as being pretty close to 7. So this figure is 4 units tall by about 7 units wide, and the two triangles fit just about perfectly, with



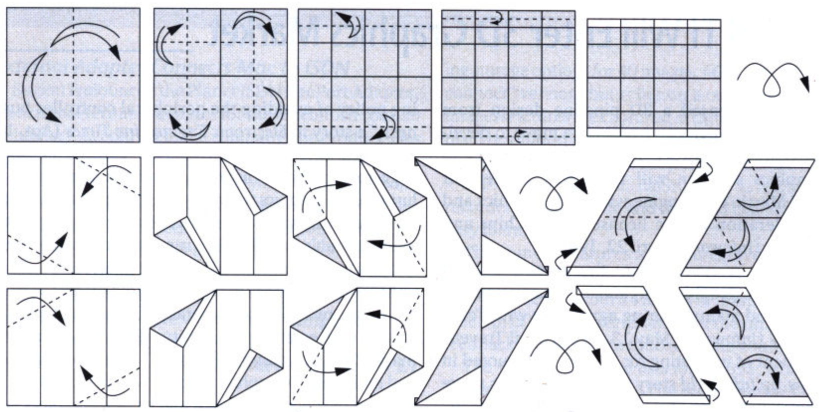
12 **(a) Dodecahedron built from triangular vertex units. (b) The triangular vertex unit. (c) Assembling triangular vertex units.**



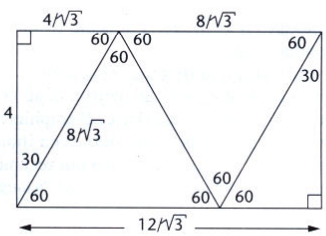
13 **Folding the triangular vertex unit.**



14 (a) Icosahedron built from four-triangle face units. (b) The four-triangle face unit. (c) Assembling two units.



15 Folding the four-triangle face unit. The second row is for the right-handed version; the third row is for the left-handed version.



16 Geometry behind the four-triangle face unit.

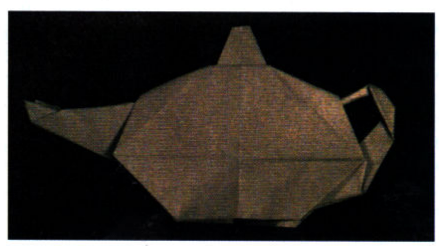
a half-triangle remaining at the left and right.

This is why we make those opening folds in Figure 15: They turn the original square into a rectangle that's 8 units wide by 7 units high (8 units wide because we're going to build four triangles, not two). If we think of the side of the square as 1 unit, then essentially we approximated the irrational value $\sqrt{3}$ as $2(1 - (1/8))$. The resulting error of about two parts in 100 is pretty close for paper-folding.

To assemble the icosahedron, continue adding pieces by inserting one of the triangular flaps into one of the triangular pockets of another unit, as shown in Figure 14c. Always bring together five triangles at a vertex, until the model is closed. You'll need five right-handed and five left-handed units.

The teapotahedron

In 1987, the geometry world was stunned when Jim Arvo and David Kirk announced in the Siggraph 87 proceedings that they had discovered a new Platonic solid, which they dubbed the teapotahedron. No discussion since then could be complete without inclusion of this new fundamental shape. Figure 17 shows my origami approximation to the teapotahedron. This version is deficient in several ways. Most noticeably, it's flat. I invite readers to create better origami constructions of the teapotahedron and to send me complete folding instructions, along with a 35mm slide (or pointer to a TIFF file). The best versions will appear in a future column. ■



17 Teapotahedron.

Acknowledgments

The units in this column were originally presented in the books by Fusè and by Gurkewitz and Arnstein. The teapotahedron was adapted from the lamp in the book by Hardin; thanks to David Kurlander for spotting it. Thanks to Jim Blinn and Jim Kajiya for pointers to Lang's publications and some other interesting corners of the mathematical origami world, and thanks to Maarten van Dantzich for testing and improving the folding diagrams.