More Origami Solids

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In my last column, I talked about the pleasures of holding a 3D model in your hand. This may have seemed a bit odd. After all, there weren't any computers involved. But I believe that the best computer graphics comes from people who bring a variety of skills to the task. One of the most important skills for anyone working with 3D graphics is a strong 3D visual imagination and 3D intuition. So this month I will continue the topic of 3D models that we can build and hold.

Archimedean solids

In the last issue we built origami models of the five classical Platonic solids: the tetrahedron, cube, octahedron, dodecahedron, and icosahedron (and, of course, the recently discovered teapotahedron). The models were all created simply by folding pieces of paper, and then putting different pieces together.

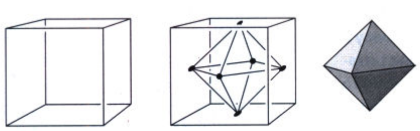
Now we'll take a look at three of the Archimedean solids. There are more of these, but we'll only look at the ones that are equal mixtures of pairs of Platonic solids.

The cube/octahedron dual

Suppose we take the cube of Figure 1a and mark the centers of the six faces with dots. Each dot has four other dots that are closest to it. If we draw a line between every dot and its nearest neighbors, we get Figure 1b. Removing the cube, we see in Figure 1c that we've created an octahedron. Its six vertices correspond exactly to the six faces of the cube.

Now imagine repeating the process on the octahedron. If you put a dot at the center of each of the eight triangular octahedron faces, then each dot has three nearest neighbors. Connecting those dots creates the outline of a cube: the eight faces of the octahedron correspond exactly to the eight vertices of the cube.

The cube and the octahedron are called *dual polyhedra*, or simply *duals*.



(a) (b) (c)

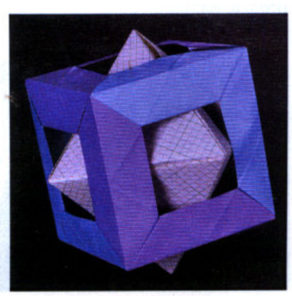
1 (a) Cube. (b) The center of each face marked by a dot connected to its four nearest neighbors. (c) Octahedron.

We can show this relationship with our origami models. I was sneaky last time because I showed how to build a framework cube and a solid octahedron. That was so that I could make the nested pair shown in Figure 2. You may have to play a little to get the sizes right, but the results are worth it. I built both models, then simply opened up the cube to place the octahedron inside.

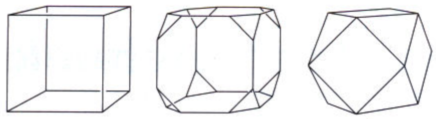
We can look at the dual construction in a slightly different way. Suppose we start with the cube of Figure 3a on the next page and begin slicing off the corners. Figure 3b shows a step along the way: the square faces become octagons, and each vertex turns into a triangle. As we deepen our slices, the triangles in the corners grow larger and larger, until they touch, as in Figure 3c. Now the faces are squares again, but they're rotated 45 degrees with respect to their previous orientation. If we continued shaving down the corners, we'd be left with an octahedron.

The shape in Figure 3c is, in some sense, halfway between a cube and an octahedron. It's an Archimedean solid known as a *cuboctahedron*. This is a celebrated shape; Buckminster Fuller had a particular fondness for this structure, which he called the *vector equilibrium*, and believed it was a basic building block of our world.

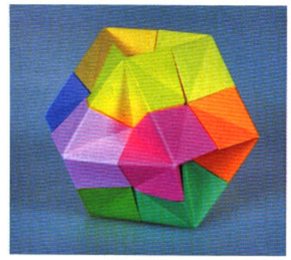
We can build a cuboctahedron with origami, as shown in Figure 4 (next page). The basic building block is a vertex unit with four radiating arms; it's a lot like the triangular piece we used last time to build the dodecahedron. Folding instructions for the *square vertex unit* are given in Figure 5. The piece is illustrated in Figure 6a, and you can see how to put them together in Figure 6b.



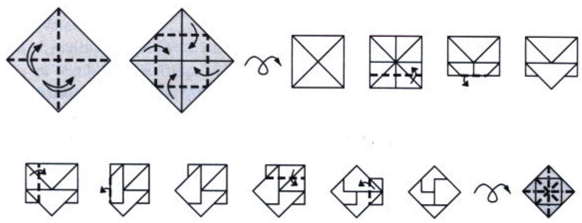
2 Nested cube and octahedron.



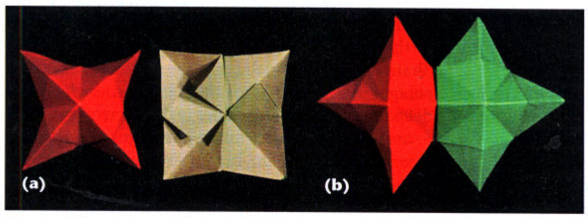
3 (a) Cube. The corners sliced (b) a quarter of the way off and (c) down to where the triangles meet in a cuboctahedron.



4 An origami cuboctahedron.



5 Folding the square vertex unit.



6 (a) Square vertex unit. (b) Assembling two units.

The tetrahedron/tetrahedron dual

Now that we've seen the basic idea behind duals, think about what the dual of the tetrahedron might be. You can use the point/face connection approach of Figure 1 or the shaving-down approach of Figure 3, but you'll reach the same answer either way: the dual of the tetrahedron is another tetrahedron! We say that the tetrahedron is *self-dual*.

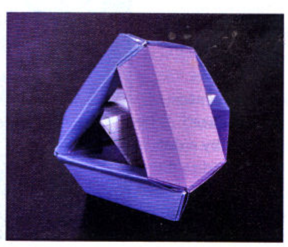
Figure 7 shows a pair of tetrahedrons, nested inside one another. Notice that just like the other dual pairs, if you file down the corners of the larger, framework tetrahedron, you'll get the one inside. The framework tetrahedron was made with a slightly altered version of the *little turtle unit*, described later under "The dodecahedron/icosahedron dual."

If you use the corner-shaving process and stop halfway, you'll reach another Archimedean solid known as the *truncated tetrahedron*, shown in Figure 8.

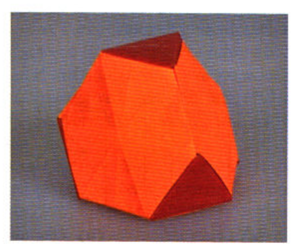
You can build this model from six hexagonal pieces of paper; three will end up forming the big hexagonal faces, and three will form the triangles. To get a hexagonal piece of paper from a square, look at Figure 9a. The basic idea is to fold a 30-60-90 triangle at each corner, and then get rid of the triangles and the flaps on the sides. The geometry behind this is shown at the far right in Figure 9b. Basically you're making sure that each side of the hexagon has unit length and meets the other sides at a 60-degree angle. The two pieces involved are shown in Figure 10.

Figure 11 shows how to build the hexagonal piece. Expect to practice on it for a bit. Folding over those flaps while getting the inside part to fold under is a snap once you see what to do, but it might take some time to get the hang of it. The step in parentheses is meant to show that you have six flaps going around at that step—you don't need to actually open up the flower. The last few steps involve tucking in three of the flaps. The result is a hexagon with pockets on three sides.

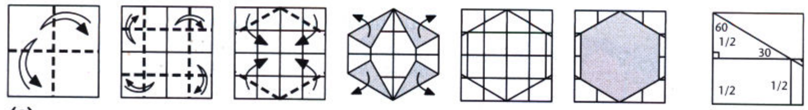
Figure 12 shows how to build the triangular piece. This is a little triangle with three flaps that go into the pockets of the hexagons. The model will hold together this way, but loosely. You might want to reinforce it with some glue or tape hidden inside.



7 A pair of nested tetrahedra.



8 An origami truncated tetrahedron, half way from one tetrahedron to another.



(a) (b)

9 (a) Creating a hexagonal from a square. (b) Geometry of part (a).

The dodecahedron/icosahedron dual

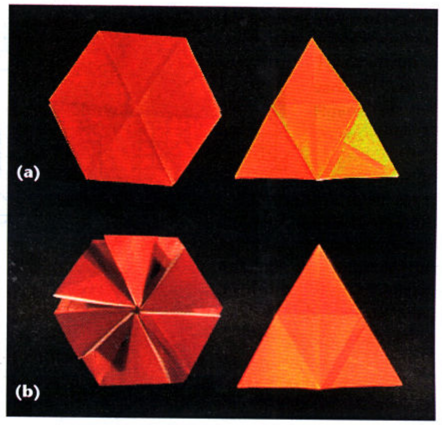
The remaining two Platonic solids are the dodecahedron and icosahedron, and as you probably expect by now, they are also duals of each other. Figure 13 shows the dodecahedron and icosahedron from last issue's column together. As with the other duals, you can see the points of the icosahedron poking out of the center of the faces of the dodecahedron.

If you whittle down the points of one of these solids, stopping when they start to touch, you reach the Archimedean solid called the *icosadodecahedron*. This is a big object. It combines the 12 pentagons of the dodecahedron and the 20 triangles of the icosahedron, for a total of 32 faces in all. It has 30 vertices and 60 edges.

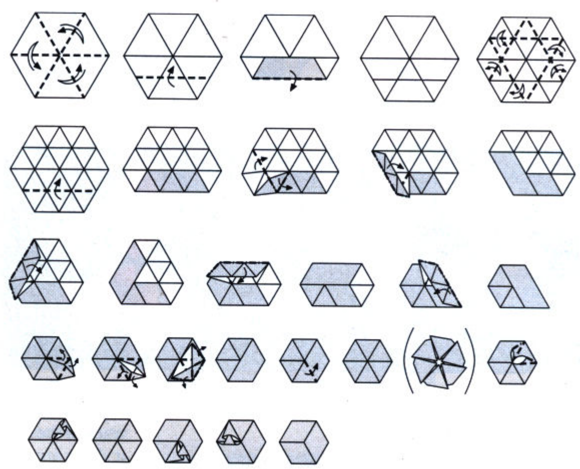
I tried a lot of approaches to building this model, but it's tricky to get something this complex to stay together; almost everything was too flimsy. The problem is that any two faces that touch are almost coplanar. If a flap sticks into a pocket, there's nothing to keep it from just slipping out again.

Finally I modified an edge-based unit called the *little turtle*, and the model hung together, more or less. Figure 14 (next page) shows my icosadodecahedron based on the modified little turtle unit. The morning I took this 12-inch diameter model to the photographer's studio, I accidentally dropped it from the incredible height of about 4 inches. One whole side of the model dented inwards, and as I tried to tease it back into a spherical shape the whole thing started to unravel. Stable, yes, but don't sneeze near it.

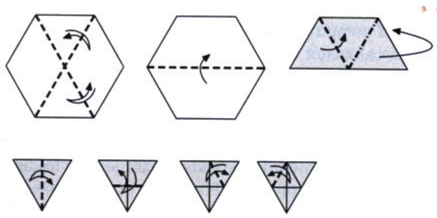
The folding diagram for the little turtle is given in Figure 15 (next page). For the variant that I used here, I opened up the triangle at the top and bottom of the unit, as shown in parentheses at the very end. Figure 16a shows the opened-up



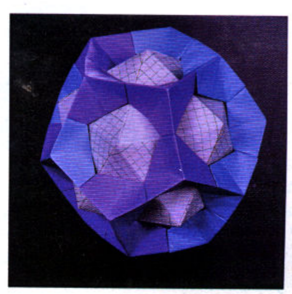
10 The pieces that make the truncated tetrahedron: (a) from the top and (b) from the bottom.



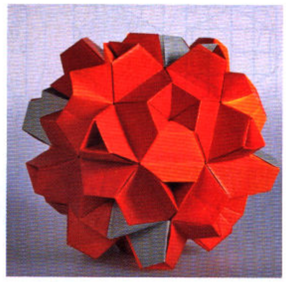
11 Folding the hexagonal face of the truncated tetrahedron.



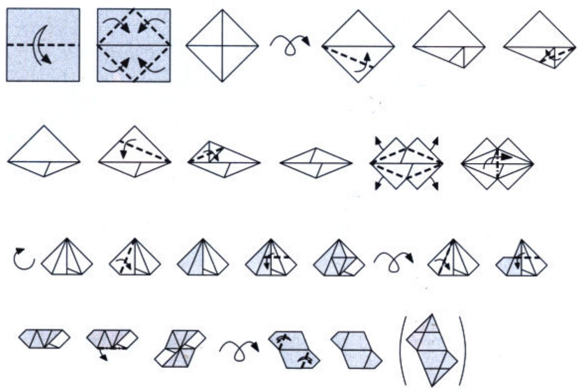
12 Folding the triangular face of the truncated tetrahedron.



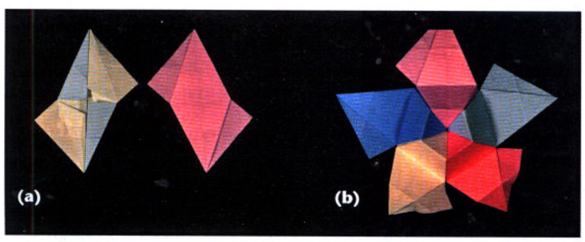
13 Nested dodecahedron (the outer shell) and icosahedron (the inner solid).



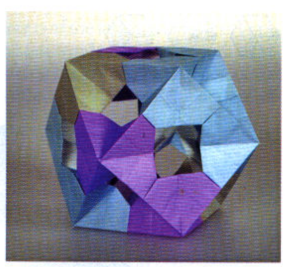
14 An origami icosadodecahedron, halfway between a dodecahedron and an icosahedron.



15 Folding the little turtle. The final figure in parentheses is the opened-up version for the icosadodecahedron.



16 (a) The little turtle. (b) Assembling turtles.



17 Another origami cuboctahedron.

little turtle; Figure 16b shows how to put them together.

Be patient as you assemble this model. It's delicate. It's also big—once you've folded the necessary 60 pieces, you'll be able to do more in your sleep.

Variations on a theme

There are many directions to generalize the techniques we've seen in these two columns. We can move on to entirely new models and classes of models or create some variations on the ones we've already built. I'm going to take the latter approach here, because most of my own understanding of how unit origami works came from playing with variations such as these.

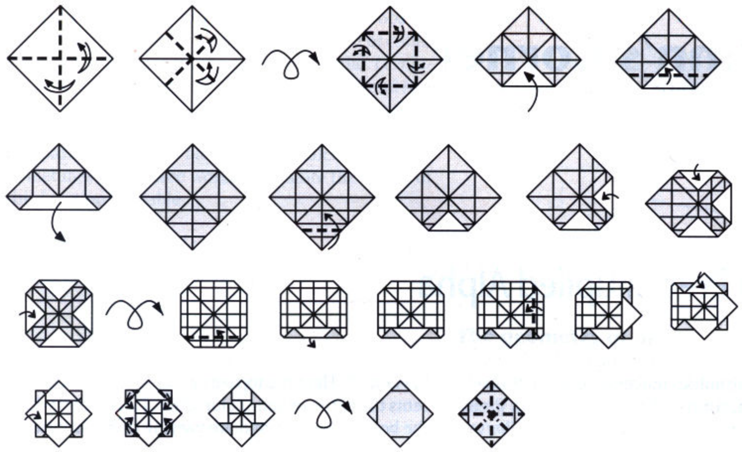
First, Figure 17 shows an opened-up version of the cuboctahedron. It takes a little more folding, but I like those windows that show through. Figure 18 shows you how to fold the piece; it looks a lot like the square vertex unit when you're done, except that the pockets don't reach all the way to the center. Figure 19a shows the pieces from above and below, and Figure 19b shows how to assemble them.

You can also play around with the folding pattern. The limitations of space here limit the variations I can show. To see more of what I've come up with, visit my Web site at http://www.microsoft.com/research/graphics/glassner/. Try cooking up your own variations on these themes; you'll find that after a while you can begin to imagine what the results will look like even as you dream up new ways of folding the paper.

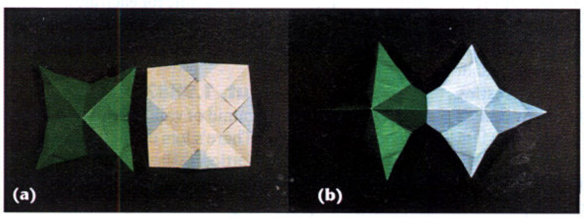
Coding it up

There are a bunch of interesting programming projects hiding within the subject of origami, and unit origami in particular. Certainly one of the most straightforward is to write a program that will read some form of origami notation and create a 3D geometry file of the result, which you can then render. Or you can use the same information to create folding diagrams. These are both very hard problems if you go into them deeply. The folding problem requires keeping track of the thickness of the paper at each fold and how it slides around (for example, if you have two layers of paper involved in a fold, the outermost layer requires more paper than the inner layer). The diagramming problem is pretty tricky, because getting the right point of view and picking the right steps to illustrate it are very personal choices.

I made the diagrams in these two columns by hand with a computer-aided drafting program. With my trusty calculator, I computed all the angles and lengths to make sure that everything lined up just where it ought to. I got very good at remembering the values for $\sqrt{3}$ and $1/\sqrt{3}$, as well as a few other key ratios. After I finished these diagrams, I learned about Maarten van Gelder's program Oridraw, which reads a text file with folding instructions and produces PostScript output. I haven't tried the program, but it's freely available from



18 Folding the variant for Figure 17.



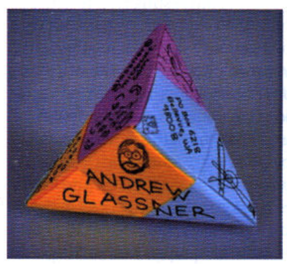
19 (a) The variant square vertex unit. (b) Assembling two units.

http://www.rug.nl/rugcis/rc/ftp/origami/programs/oridraw/.menu.html if you want to give it a whirl. A whole bunch of other origami-related programs can be found there as well.

Moving on

Unit origami is only a few years old, and it's growing quickly. I encourage you to build some of the models in this column and cook up some variations of your own, using decorated paper or mixing modules up. I used a variation on the tetrahedral units to notify friends the last time I moved. They received three colored squares with writing and a page of folding instructions to create the 3D moving card in Figure 20.

Happy folding! ■



20 A moving card built from unit origami.

Acknowledgments

The units in this column were originally presented in the books by Fusè, and Gurkewitz and Arnstein. Thanks to Alvy Ray Smith for encouragement in pursuing origami polyhedra, Jim Blinn for locating a cool book on the subject, and Bobby Bodenheimer for finding a copy of Lang's CalTech article on origami.

Further reading

The basics of unit origami were first put forth in *Unit Origami* by Fusè (Japan Pubs). *3-D Geometric Origami* by Rona Gurkewitz and Bennett Arnstein (Dover) is a little light on discussion but presents a wide selection of units. For more traditional origami, I recommend *The Complete Book of Origami* by Lang (Dover Pubs). For more fun with paper, look at *Paper Dinosaurs* by David Hawcock (Sterling Publishing) and *Paper Capers* by Jack Botermans (Henry Holt).

Two of Robert Lang's technical articles on origami are particularly interesting. "Mathematical Algorithms for Origami Design" appeared in *Symmetry: Culture and Science*, Vol. 5, No. 2, 1994, pp. 115-152. "Origami: Complexity Increasing" appeared in the CalTech quarterly magazine *Engineering & Science*, Vol. LII, No. 2, Winter 1989, pp. 16-23.