Net Results

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Poker players fold. Film directors cut. Recording engineers tape. In this column we'll do all three. In *IEEE CG&A's* July and September 1996 issues I discussed how to fold interesting polyhedra using origami techniques. The idea there was to build 3D shapes from square or rectangular pieces of paper by simply folding them. This time I'll throw out that restriction, and use scissors and glue as liberally as folds. Actually, we'll only use scissors to cut out the initial shape. After that, just fold and glue.

My motivation is the same as before: A good 3D visual imagination is valuable for anyone working in 3D graphics. An excellent way to develop that imagination is to build models and hold them, turn them, and study them. Building them yourself, you benefit from the physical process of assembly—you really feel how the pieces fit together, which deepens your understanding of the pieces and their relationships. I find that building models helps me keep my 3D visual eye tuned up and active.

We'll begin, as before, with the Platonic solids. I'll show an unfolded shape, such as a tetrahedron made of four equilateral triangles, and include small assembly flaps. Each solid line, including the ones between the triangles and the flaps, indicates a mountain fold—this means that you should fold the pieces of paper on both sides of the line away from you so that the fold itself rises toward you. The dashed lines are valley folds, which recede from you.

Construction tips

To build these models, I recommend card stock—not as heavy as thick cardboard, but stiff enough to resist buckling. Something a little thicker than a standard business card should do the trick. You can buy this sort of paper at any art or stationery store.

You'll also want to work larger than the diagrams. I recommend two to three inches for most edges. It's usually advantageous to work as large as possible—larger pieces cover up minor errors in measuring, folding, and cutting, and are usually more satisfying to hold and manipulate. I generally use one of three different enlargement methods, depending on the diagram's complexity. The first is to draw the diagram directly onto the card stock, measuring it out from the original diagram's geometry.

When I don't want to measure right on the surface, I use two other approaches. Both begin by creating a full-size drawing of the diagram on a big piece of paper (or a few pieces of paper taped together), either by measuring it out or using an enlarging photocopier on a preexisting diagram. One way to transfer this full-size drawing to the card is to first scribble on the back of the paper with a soft-lead pencil. Then tape the paper over the card stock and draw over the diagram with a sharp pencil; this transfers the lead from the back of the paper onto the board, leaving a light line for cutting and scoring. For some diagrams, I tape the paper down and use a pin to make a small holes through the paper onto the board underneath, usually at intersection points and external corners. Then I make creases and cuts with the help of a straight edge lined up to the pinholes.

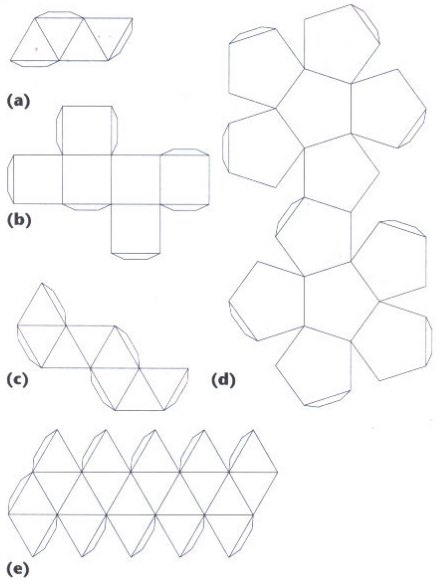
To prepare for folds, I recommend scoring the card by running a blunt butter knife over the fold lines. You can also bear down with a ballpoint pen—you can use one without ink if you don't want to leave a mark.

Folding up the models may take some trial and error, particularly those at the end of the article. I recommend artist's layout tape for holding the pieces together while you play with them. This tape uses the same tacky but removable adhesive that's on the back of those yellow sticky notes.

You can use any regular glue to assemble the flaps. But be careful if you're decorating your model using paper glued over the thicker card. You'll want to remove the paper from the flaps and their destinations so that you're gluing the thick card directly to itself. If you don't want to use glue, you can use regular sticky tape. You can place the tape inside the model if you don't want it to show, but that becomes tricky to apply. Alternatively, you can place the tape on the outside. I find this is the best way to play with the models, because it lets you take them apart easily to study the relationship between the 2D diagram and the 3D model. When you want to make them permanent, glue's the way to go.

Nets and efficiency

The unfolded tetrahedron in Figure 1a (on the next page) is the simplest of the Platonic solids. The standard hierarchy then continues with the cube, octahedron, dodecahedron, and icosahedron. Figure 1 shows the unfolded diagrams, or nets, for these solids.



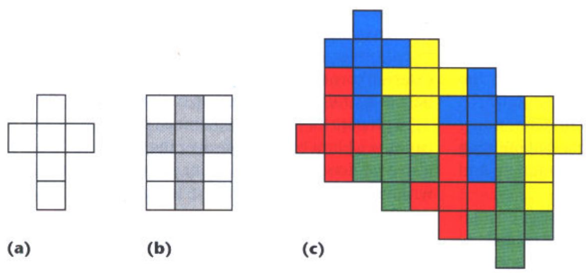
1 Nets for Platonic solids: (a) tetrahedron, (b) cube, (c) octahedron, (d) dodecahedron, and (e) icosahedron.

If you're just making a few little models for fun, then it's fine to draw the net somewhere on a piece of paper and cut it out. But suppose you need to make thousands or millions of models—one per rectangular piece of paper. Can different nets give you more or less efficient use of the area?

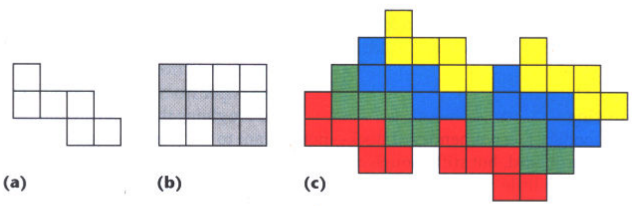
Figure 2a shows the most traditional net for the cube. When considering efficiency, we'll ignore the flaps. Assuming an edge length of 1 for each square, the net has an area of 6 and sits in a rectangle of area 12, as in Figure 2b. So the efficiency equals 50 percent. If efficiency were really an issue, you'd want to use that empty area within the rectangle. Happily, just as 3D cubes fill space, these 2D nets tile the plane, as in Figure 2c. So the efficiency goes up to 100 percent in the middle of the paper where they tile, and you only lose the little bits of trim at the boundaries of the sheet.

Figure 3 shows another net for the cube. Although the pieces have been moved around quite a bit, this net also has an efficiency of 50 percent and tiles the plane. Figure 4 shows yet another net that tiles the plane and also achieves an efficiency of 60 percent. I don't know of a net for the cube with better than 60 percent efficiency.

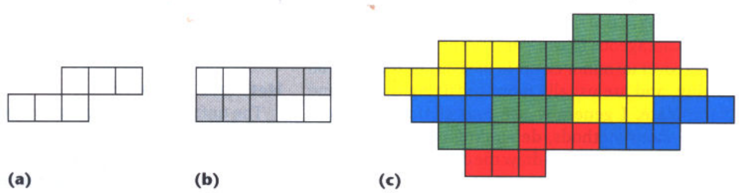
Figure 5 presents the net for an octahedron, with an efficiency of 50 percent. It also tiles. A little transposition of the net again yields 50 percent, as in Figure 6. Finally,



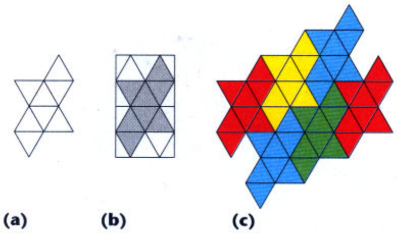
2 (a) A net for the cube. (b) Showing 50 percent efficiency. (c) Tiling the net.



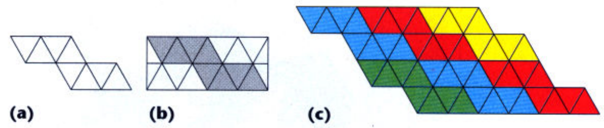
3 (a) A second net for the cube. (b) Showing 50 percent efficiency. (c) Tiling the net.



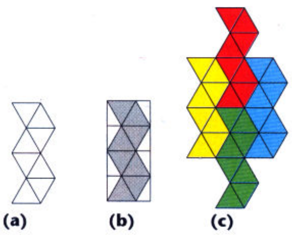
4 (a) A third net for the cube. (b) Showing 60 percent efficiency. (c) Tiling the net.



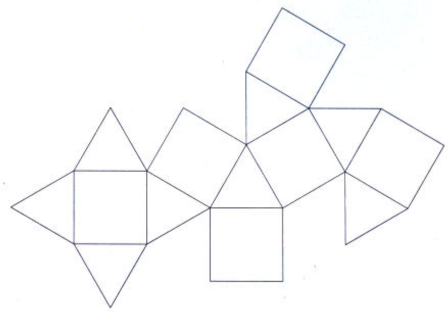
5 (a) A net for the octahedron. (b) Showing 50 percent efficiency. (c) Tiling the net.



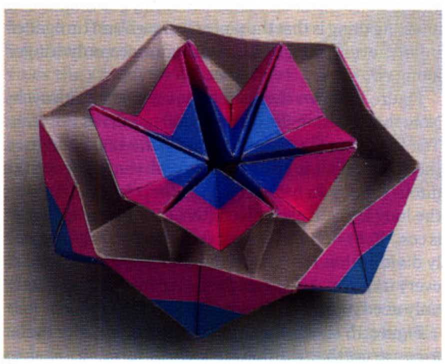
6 (a) A second net for the octahedron. (b) Showing 50 percent efficiency. (c) Tiling the net.



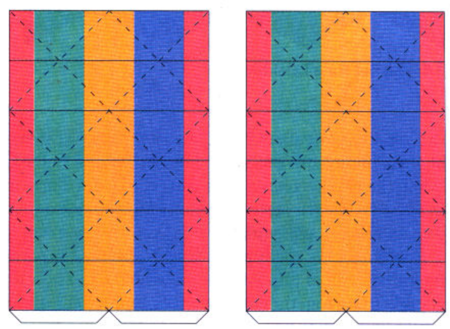
7 (a) A third net for the octahedron. (b) Showing 67 percent efficiency. (c) Tiling the net.



8 The net for the cuboctahedron.



9 The polyhedral flower.



10 The net for a polyhedral flower. The two pieces are glued to one another to make a long cylinder, colored side outward.

11 A piece of the flower ready for assembly.

Figure 7 shows the best efficiency of all—67 percent.

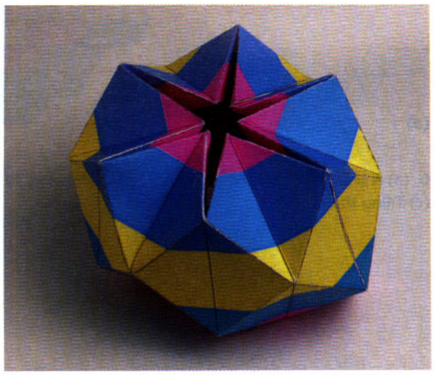
Although each net has eight equilateral triangles, they also have the necessary connectivity. When playing with nets, make sure you don't accidentally move the pieces around so that you're unable to fold the desired shape.

Finally, in Figure 8, I can't resist providing the net for the Archimedean solid called the cuboctahedron, which was Buckminster Fuller's favorite shape.

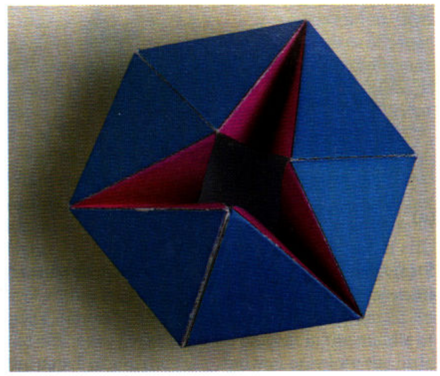
Flowering polygons

So far we've folded up flat diagrams into static 3D shapes. You can build all kinds of dynamic 3D models from very simple nets. One of my favorites is called simply the polyhedral flower. Figure 9 shows the flower, with the petals inside just rising and starting to spread.

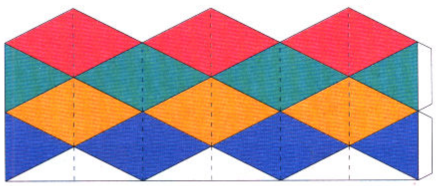
Figure 10 shows the nets for the flower—you'll need both of these. Figure 10 also shows suggested coloring. If you use this scheme, use the same coloring for each of the two pieces. Cut them out carefully, and score the mountain and valley folds very precisely; accuracy will pay off in a more stable and beautiful model. Figure 11



12 Folding the flower.



13 A partly open kaleidocycle.



14 The net for a kaleidocycle.

shows how a piece should look when it's ready for assembly (the photos use a slightly different color scheme). When the two nets have been scored, glue the flaps at the bottom of each piece to the top of the other, so that you create a cylinder with the colored side facing out.

Now comes the tricky part—but it's also the fun part. It took me a bit of fumbling before I figured out how to get this thing together, but when assembled it's quite lovely. I'll describe how I finally managed it, but there's nothing quite like holding the model in your hands and seeing it. Expect to experiment a bit before you get there.

Most importantly, you must have good folds. Holding the cylinder so that its axis runs vertically, pull together the triangle pairs on the top until they begin to come together, causing the diamond shapes on the outside to fold inward. Figure 12 shows how this should look. You can use artist's temporary tape to hold the folds together. Now do exactly the same thing on the other side. There's no phase shift here—that is, the model is exactly symmetrical (except for color) on both sides of an imaginary plane that cuts perpendicular to the axis of the cylinder. When you're done, you'll have something like a polygonal donut.

Now comes the easy part. Take off the tape and let the top part flop open a little while keeping the bottom points together. Pull out on the red points at the top while pushing in on the yellow points at the bottom. This will cause the whole ring to rotate around itself. The top part will open up into a hole, the points from below will pop up, and the points will begin to spread apart. If you continue to rotate the ring, it will settle into a stable position with the center part forming a fountain of color in the midst of an enclosing cup, as in Figure 9. You can continue to turn the ring around and around, causing the flower to fold into itself and then bloom again.

Kaleidocycles

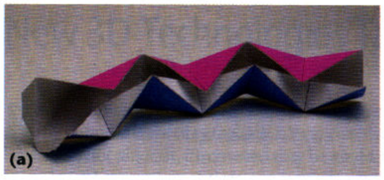
The kaleidocycle is one of the most interesting dynamic polyhedral models. A ring of nonregular tetrahedra, it rotates like the flower but doesn't open up. Figure 13 shows a kaleidocycle. To really appreciate it, you need to build one and turn it in your hands. The amazing thing is that the pieces just turn and turn indefinitely, even though the tetrahedra are only hinged along two edges.

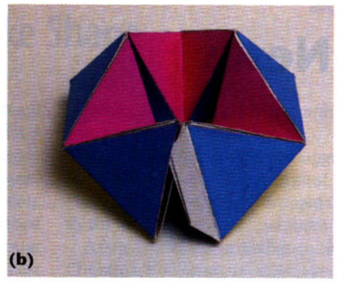
Figure 14 shows the net for a hexagonal kaleidocycle. Note that the triangles are not equilateral, but isosceles. The long altitude of the triangle is the same length as the short side it is erected from. You can work out the dimensions from basic trig; if the short leg has length 1, the long legs are each of length $\sqrt{1.25}$. The acute angle is $\cos^{-1}(0.6) \approx 53.13$ degrees; the other two angles equally divide what remains of the 180 degrees allotted to every triangle—they're each about 63.43 degrees. You'll only need one copy of the net to build the model.

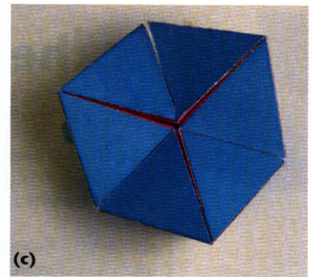
Figure 15 shows the folding process. It's a lot easier to construct than the flower. Begin by forming a ring, so that the top colored triangles overlap with the uncolored ones at the bottom. Put glue on the bottom flaps and adhere them to the bottom of the upper triangles. Then put glue on the flaps at the end and tuck them into the hole at the other end, forming a ring. As you turn the ring, it will stay together as a single stable structure, showing you four distinct images as you turn the pieces. It's fascinating to watch the shapes move around one another, seemingly unfolding forever.

If you're interested in decorating your kaleidocycle, Figure 16 shows a schematic of how the pieces connect when they form images. The four colors make up the four images, and the letters indicate which arrows join together. If you get continuity across the arrows, then you'll be able to form a separate image from each of the four sets of six triangles.

For an even greater challenge, you can try to get continuity around the ring as well as within each image—



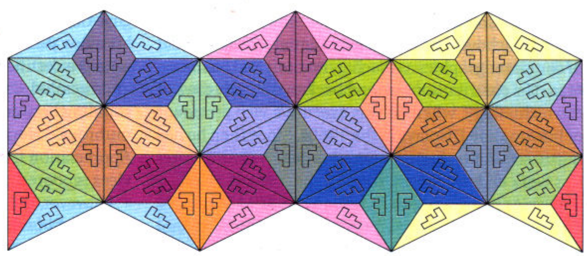




15 Constructing the kaleidocycle. (a) Form a tube and glue the uncolored pieces under the colored ones. (b) Tuck the end flaps into the open slot. (c) The assembled kaleidocycle.

16 The connectivity relations between the triangles for the kaleidocycle's four images. Arrows with the same letter are adjacent in the folded model. The white arrows indicate edges that share continuity during rotation.

this is indicated by the white arrows in Figure 16. Figure 17 shows the full-blown version of this, which provides the symmetry constraints to pull off the more complicated continuity. Readers of this column will recognize the "F" motif as my favorite way to indicate an oriented tile. All tiles with the same color here should have the same internal patterns. Notice that although there are 72 tiles, only 24 different designs are used. There are 12 pairs—the ones that run vertically in the figure. These triangles remain adjacent in the unfolding pattern because they're adjacent in the net. There are also 12 quadruplets, each arranged in what looks to me like butterfly wings. Technically, these little symmetry markings are sufficient but not necessary. That is, they will do the job, but they're overkill. You really only need continuity of design across the edge. As long as they mesh where they touch, the contents of every tile can be different.



17 The "F" motif indicates the tile's orientation. Tiles with the same color contain the same decoration. There are 12 pairs and 12 quadruplets in the 72 tiles.

If you use Figure 17 as the decoration for a kaleidocycle, don't get confused. When I folded this to make sure I drew the figure correctly, I started folding along all the solid lines, which is unnecessary and makes for a very floppy model. These lines are just the boundaries around each tile, not the folding pattern—that's in Figure 14.

Happy cutting, folding, and taping! ■

Further Reading

A great book for making all kinds of fixed models is *Polyhedron Models* by Magnus Wenninger (Cambridge University Press). I also highly recommend his other books.

I first saw the polygonal flower in a little book called *Mathematical Curiosities 1* by Gerald Jenkins and Anne Wild (Tarquin Publications). It contains nine interesting little math curiosities, each illustrated with a little model. The book contains full-color pages that you can cut out and score to build the models. Figure 8 uses their five-striped decorating scheme.

Doris Schnattschneider and Wallace Walker present a beautiful decoration of kaleidocycles in their book *M.C. Escher Kaleidocycles*. Besides giving a nice description of the symmetries in Escher's tiled drawings, they show how to adapt his drawings to different kaleidocycles. The book contains high-quality, prescored, die-cut nets for building your own models.