Andrew Glassner's Notebook

Inside Moiré Patterns

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Moiré patterns are very cool. They're produced by the interaction of two overlaid patterns. You saw moiré effects any time you looked through two pieces of fine-mesh screen and noticed the broad dark bands that appear as the meshes moved or as you moved your head.

Created deliberately, moiré patterns are an inexpensive special effect. Some children's books consist of printed pages plus a transparent sheet with a black pattern on it. By waving the transparent sheet over the book's pages, you can make waterfalls flow and clouds float by.

Moiré patterns also creep in where they're not wanted. When you scan a photograph, the dot pattern used originally to print the image can interact with the digitizing dot pattern used by the scanner, sometimes resulting in bands or blocks of light and dark across the image. Badly registered color screens can interact in a similar way, so what ought to print as a smooth field of color looks something like a busy plaid.

Moiré patterns also have a practical side. The field of moiré interferometry uses these patterns to measure very small displacements in surfaces and thin materials.

The principles behind moiré patterns aren't very complicated. With a little geometry, it's easy to understand where they come from and how to create and control them.

Gratings

To demonstrate moiré effects, I'll use a very simple set of patterns: sheets of parallel lines, called *gratings* (see Figure 1). At the end of the column I'll show examples of grids, dots, and circles creating moiré patterns.

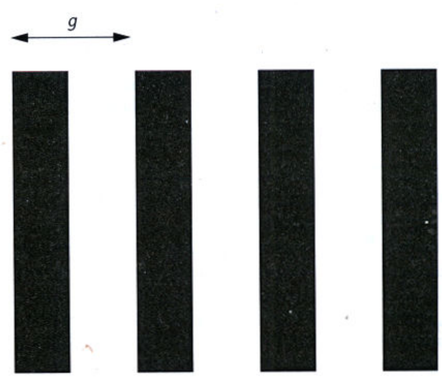
In a grating, the black lines all have the same width, as do the white spaces between them. We can think of one adjacent pair of black and white bars as defining the fundamental region of the pattern—rubber-stamping that pair of bars side by side creates the grating.

A grating is characterized by two numbers. The *pitch*, usually denoted by g, describes how closely packed the lines are. Now, because these are real lines and not mathematical abstractions, they have some width. Referring to the vertical lines of Figure 1, I find it convenient to think of the pitch as the distance between the left-hand edges of neighboring black bars. The *ratio*, usually written R, is the fraction formed by the width of the black bar divided by the width of the white bar. A ratio of 1 means that the two bars have equal width.

Extension

Let's begin by creating two gratings, each with a ratio of 1. The first grating, p_1, has more lines per inch than the second grating, p_2, so pitch g_1 is smaller than g_2. We'll pick the pitches to be nearly similar, say within 5 percent of each other. Let's overlap the two gratings, as in Figure 2, so that the lines are parallel. This kind of alignment is called *extension*.

You can see that a new pattern emerges, consisting of black bars that seem to represent a much wider grating. These bars, called *moiré fringes*, are, in fact, equidistant. The fringes come about because the two gratings "beat" against one another. What is the pitch G of this new pattern?



1 A grating is a set of alternating black and white bars. All the black bars are the same width, as are all the white bars. The pitch g is the width of one pair of bars.

(a) (b) (c)

2 (a) Grating p_1. (b) Grating p_2. (c) A moiré pattern created by pure extension—simply overlaying (a) and (b).

Figure 3 provides a way of answering this question. Here I've drawn a simple intensity profile of the bars as waves—a value of 0 means black, and 1 means white. At the left edge of the figure, the two gratings are aligned. But after just one cycle, they've slipped a little—the finer (or more compact) grating p_1 has already started into its second white bar before the coarser grating p_2 has finished with its first black bar. As we scan the pattern from left to right, we see that p_2 falls farther and farther behind p_1, until eventually it falls one complete cycle behind and the two patterns align again. It's just like one runner lapping another around a track.

How does this create the moiré pattern? Consider that the page you're reading this on arrives from the manufacturer completely white. The printing process deposits black ink to block the ambient light reflecting off the page and into your eye. So anywhere we've printed black (due to either grating), the page will be black. Viewed the other way, it's only white where both gratings are white. We can find the white regions by simply taking the logical AND of gratings p_1 and p_2, producing the new pattern p_M. Notice that the black regions get wider as you scan from left to right, then narrower again. This composite pattern returns to where it started after one full cycle, which takes up a distance G.

The cycle repeats when p_1 takes one more cycle than p_2. In symbols,

$$G = n\,g_2 = (n+1)\,g_1$$

By noting that $1/g_2 = n/G$ and $1/g_1 = (n+1)/G$, we can eliminate n and write

$$\frac{1}{G} = \frac{1}{g_1} - \frac{1}{g_2}$$

Often it's convenient to speak of frequency rather than length of the cycles (or waves). Conventionally, frequency f is the inverse of wavelength. Suppose we have some nice system of units so that for any given grating pitch g, the corresponding frequency f is simply $1/g$. Then we could write this equation equivalently as

$$F = f_1 - f_2$$

The frequency of the repeating black bars is simply the difference in the frequencies of the two gratings.

Amplification

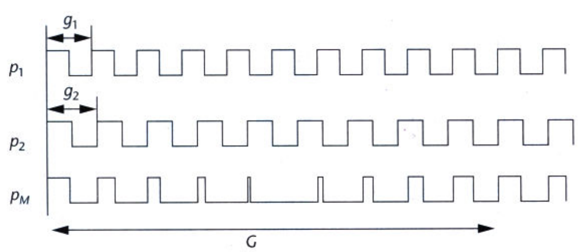
Now suppose we slide over one of the gratings with respect to the other. What happens to the black bars? Well, we know that the distance between them doesn't change, since that depends only on the difference in the pitches (or the frequencies). But does the position of the bars move?

Suppose that grating p_2 moves by some integer multiple of its pitch g_2. Then everything lines up exactly as before, and there's no change. But suppose that p_2 moves to the right by $g_2/2$. Intuitively, we would expect the fringes to move half their separation, or $G/2$. Similarly, if we move p_2 to the right by $g_2/4$, we'd expect the bars to move to the right by $G/4$. This is exactly what happens, as illustrated by Figure 4.

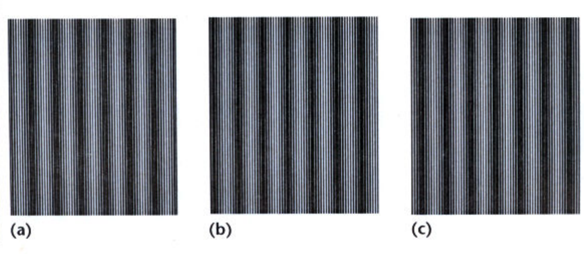
This phenomenon of *fringe amplification* comes about simply because everything is linear—moving the grating some distance between its bars moves the moiré fringes by a similar amount. In symbols, if we move grating p_2 by a distance δ, the moiré pattern moves by a distance Δ, which is simply δ scaled by the ratios of the two patterns:

$$\Delta = \delta\left(\frac{G}{g_2}\right)$$

The term amplification here refers to the large movement of the fringes in response to the smaller movement of the underlying grating.



3 The two gratings of Figure 2 shown in schematic form. The period of the resulting moiré pattern is G.



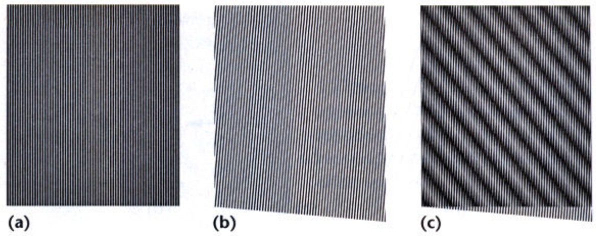
4 The gratings of Figure 2. (a) The gratings are aligned left. (b) Grating p2 has been translated a distance g2/4. (c) Grating p2 has been translated a distance g2/2.

Rotation

Let's return to our original two gratings, p_1 and p_2, with equal ratios and slightly different pitches. This time we'll superimpose them so that they're slightly tilted with respect to each other. To make things easy, we'll assume that the lines of p_1 are vertical, and the lines of p_2 are nearly vertical—say within about 10 degrees of vertical. What happens?

Figure 5 shows the answer. The fringes reappear as black bars with spikes on the side, separated by white rhombuses. If you squint, the individual lines of the gratings disappear, and all you see are the thick bars of the fringes themselves. In practice, this often happens—the gratings are all but invisible, but the moiré fringes reveal their relationship.

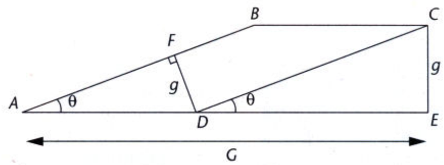
What is the pitch of these new fringes, created by pure rotation? The geometry of this pattern appears in close-up in Figure 6. Each white space is a parallelogram with dimensions dictated by the pitches and ratios of the gratings and the angle θ between them. We will assume that both gratings have the same pitch g and that the lines are very thin, so the black-to-white ratio R is nearly zero.



5 **(a) A vertical grating. (b) A slanted grating. (c) The moiré fringes resulting from the overlay of (a) and (b).**

The parallelogram $ABCD$ is the white region; E is the point of intersection of line AD and the line through C perpendicular to BC. We know that distance CE is the pitch g. We know that $\tan\theta = (CE)/(DE)$, so distance $DE = g/\tan\theta$. If we erect a perpendicular on line AB such that it passes through D, the distance FD is also g (because it's the space created by the second grating). Since AD is the hypotenuse of right triangle ADF, we can see $g = AD \sin\theta$, or $AD = g/\sin\theta$. So the pitch of the fringe, G, is given by

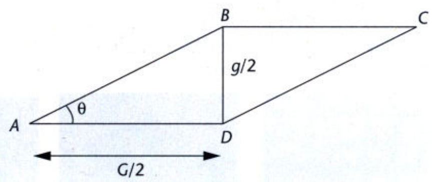
$$G=(AD)+(DE)$$
$$=\frac{g}{\sin\theta}+\frac{g}{\tan\theta}$$
$$=g\left(\frac{1}{\sin\theta}+\frac{\cos\theta}{\sin\theta}\right)$$
$$=g\left(1+\cos\theta\right)/\sin\theta$$



6 **The geometry of a white parallelogram from Figure 5.**

We assumed that the lines were very thin, but that doesn't change the analysis. If the lines get thicker, the parallelogram formed by the white space will get smaller, but the pitch G won't change.

However, if we're willing to assume that the angle θ is small and that the ratio is 1, we can make some simplifications. Figure 7 shows the reduced geometry. Since θ is small, we can pretend that the line BD is almost vertical (that is, perpendicular to both AD and BC). Then we can observe that $BD = g/2$ and $AD = G/2$, and write



7 **Simplified geometry of Figure 6 for small angles.**

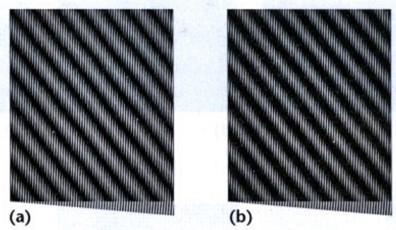
$$\tan\theta = BD/AD = (g/2)/(G/2) = g/G$$

For small angles, $\tan\theta \approx \theta$. Using this and solving for G, we find $G = g/\theta$, which is the standard formula for the moiré pitch G for small rotation θ. In terms of frequency, $F = f\,\theta$, which tells us that the fringes get closer together (and thus harder to resolve) as the angle gets larger. This argues for using smaller angles. But as we decrease the angle, the fringes can get hard to discern. In practice, picking the right angle and the right pitches of the gratings is critical to getting useful results.

Fringe sharpening

The fringes in Figure 5 are clear but quite wide. It might be easier to locate their centers if the fringes were narrower. We can make this happen with a technique called *fringe sharpening*.

It's actually quite simple to sharpen the fringes. Think about their appearance for a second: the black fringe arises when the two gratings lie atop one another. The long spikes are caused by the angled grating cutting across the white gaps in the vertical grating. If we keep the same pitch in the angled grating but make the black bars thinner (that is, we decrease the ratio), we will decrease the width of the spikes. At the same time, increasing the thickness (or ratio) of the vertical grating increases the area of the overlaps, strengthening the fringes.



8 **Fringe sharpening. (a) Original ratios. (b) Reciprocal ratios.**

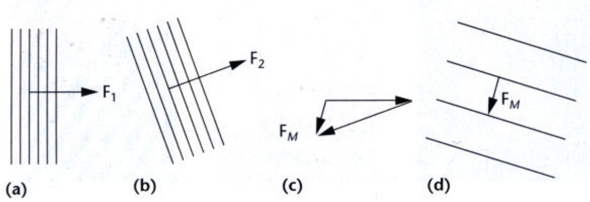
The sweet spot occurs when the ratios are reciprocals of each other:

$$R_2 = 1/R_1$$

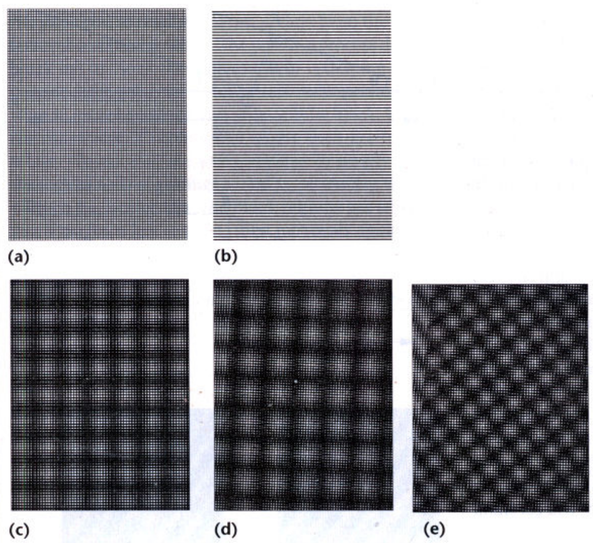
Figure 8 shows an example of fringe sharpening in the case of pure rotation.

Vector addition

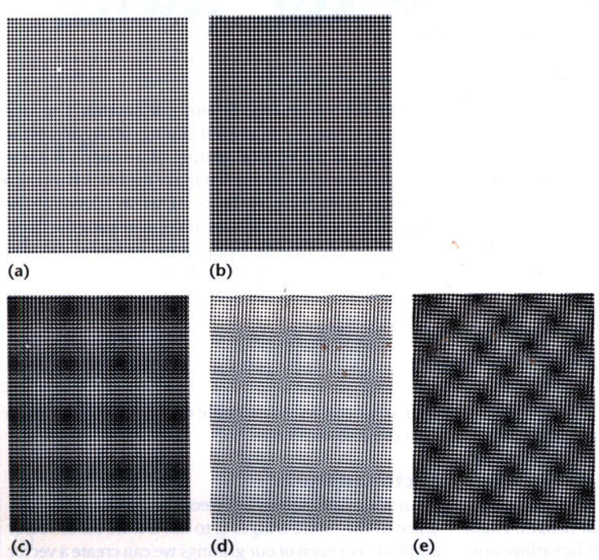
Another way to find the direction and interfringe distance of the moiré fringes is to use a little bit of vector addition. For each of our gratings we can create a vector



9 (a) Grating p_1 and its vector F_1. (b) Grating p_2 and its vector F_2. (c) Vector addition: $F_M = F_1 - F_2$. (d) The resulting moiré grating generated by F_M.



10 (a) Grid p_1. (b) Grid p_2. (c) Pure extension of the two overlaid grids. (d) Pure rotation: a copy of p_1 has been placed over itself, rotated by 5 degrees. (e) Combined rotation and extension: p_2 rotated 5 degrees and placed over p_1.



11 The same sequence as Figure 10, only using grids of dots.

whose magnitude equals the frequency and whose direction is perpendicular to the lines of the grating. Figure 9 shows the idea, where vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ are built from gratings p_1 and p_2.

The resulting moiré fringes are characterized simply by a new vector, $\mathbf{F}_M$, given by

$$\mathbf{F}_M = \mathbf{F}_1 - \mathbf{F}_2$$

Note that this short vector indicates a larger gap between the fringes than between black bars in the original gratings, which is the same phenomenon we've been seeing all along.

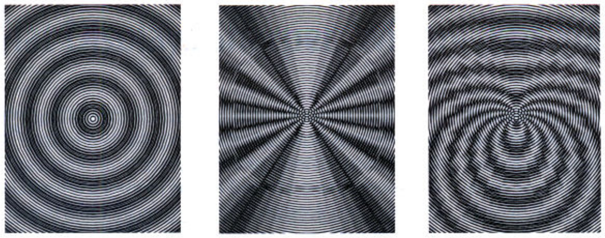
Grids, dots, and circles

So far, we've looked only at gratings, which are sheets of parallel lines. We've all seen moiré patterns created by wire meshes, which are simply two sets of gratings at right angles to one another. Everything that we've done for gratings can be used to predict the interaction of two meshes (or grids), simply by superimposing the results. But the fringes are much more interesting to look at!

Figure 10 shows the result of two grids using pure extension, pure rotation, and combined extension and rotation. The two independent sets of gratings combine to produce the diamond fringes.

At the start of this column I mentioned that moiré patterns can pop up when the dot pattern of a scanner interacts with the dot pattern of a digitized photograph. Figure 11 shows our three canonical cases for two square patterns of dots. Of course, the dots are nothing but the points of intersection of the grid lines in the mesh case, so we would expect (and find) a lot of similarity between Figures 10 and 11.

A twist comes in when we start to work with nonlinear elements. Figure 12 shows the interaction of two circular gratings; here the pitch equals the radial distance between the insides of two successive bands, or annuli. In Figure 12a I've overlapped two circular gratings with near-equal pitches over a common center. You can see the moiré fringes come and go just like the linear fringes in Figure 2. In Figure 12b I've used two circular patterns of the same pitch, but I've moved one of



(a) (b) (c)

12 **Concentric circles. (a) Different pitches (pure extension). (b) Identical pitches, but offset centers. (c) Different pitches and offset centers.**

the circles with respect to the other. Finally, in Figure 12c I've combined the two effects, so the displaced second pattern also has a slightly different pitch. A very interesting set of curves arises. I suspect that a nice explicit formula exists for this family of curves, but I don't know what it is.

Wrapping up

We've only scratched the surface of moiré patterns here. It's interesting to look at the interaction of linear and circular patterns, elliptical patterns, and patterns of any other kind of geometry you'd like to invent. Once you get familiar with how to predict the effects of different patterns, you'll find that it's easy to create new ones on demand. You may also be able to cook up some strategies for removing moiré patterns in cases where they creep in uninvited. ■

Acknowledgments

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