# Andrew Glassner's Notebook

## Celtic Knots, Part 2 _____

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In my last column I talked about Celtic knotwork. This name covers a wide variety of beautiful 2D patterns created by weaving one or more bands in perfect over-and-under harmony. In this column I'll pick up where I left off and talk about some of the techniques that I've discovered for creating and manipulating knotwork.

### Keeping the band together

Let's begin with the grid-oriented knots we saw last time. Figure 1 shows the underlying structure of a typical knot—a skeleton drawn inside a connected grid of squares. Inside each square you see two pieces of the skeleton, each joining opposites sides of the square, one passing over the other in the process. The over-and-under pattern alternates perfectly along the path of the skeleton. (For more information on knots and skeletons, see last issue's column.) You can think of each square as a little square tile with a pattern drawn upon it.

One way to add interest to this knot is to replace one of the squares containing an intersection with a square containing a pair of band segments that join up neighboring sides, as in Figure 2. I call a tile containing an intersection an *X-tile,* and one containing two quarter circles a *T-tile* (the T is in honor of Father Sébastien Truchet, 1657-1729, who first investigated tiles of this form—in addition to his work in dynamics, fluid flow, and typography). You can place a T-tile in two ways. If you label the sides with compass directions, then the north edge can connect to either the east or west edge.

Figure 3 shows the result of each of these choices in the knot of Figure 1. Notice that the choice in Figure 3a maintains the single continuous band, while the choice in Figure 3b leaves us with two distinct bands.

It would be nice to predict what was going to happen before making the replacement. Sometimes when working with a design I want to preserve the band count, and sometimes I want to deliberately increase or decrease it. Replacing X-tiles with T-tiles either preserves the band count or bumps it up by one. Running the replacement the other way similarly preserves the count or decreases it.
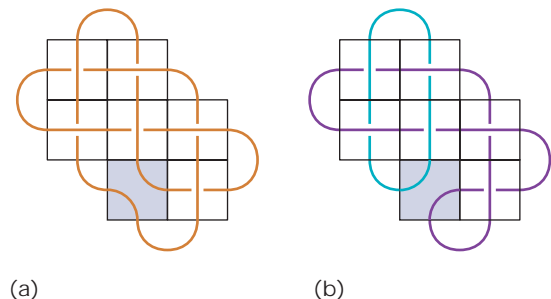
The brute-force approach to building up a design is to replace one of the tiles and then manually run around the pattern, marking each piece of band as you traverse it.
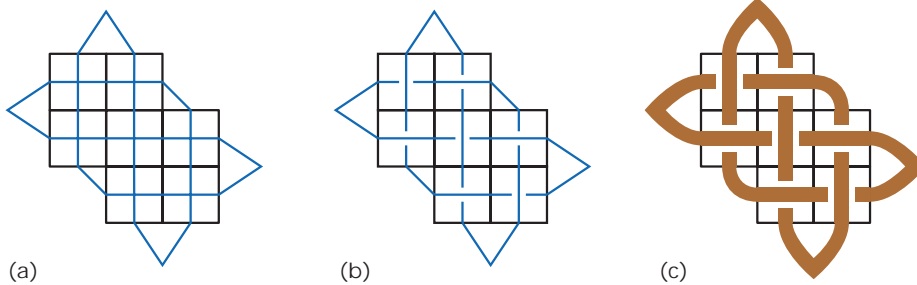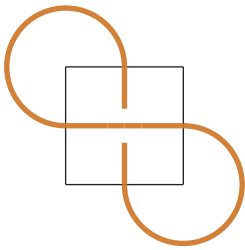


**1** Developing a Celtic knot. (a) The basic skeleton. (b) The skeleton is broken where it passes under itself. (c) The final band.
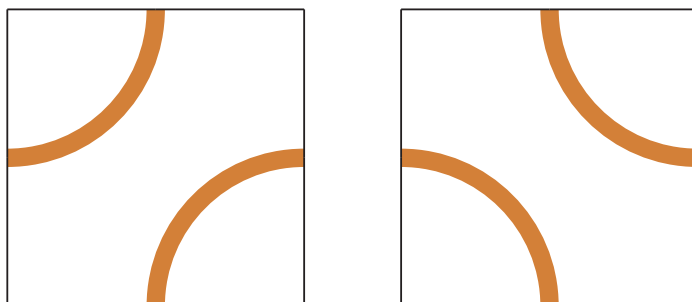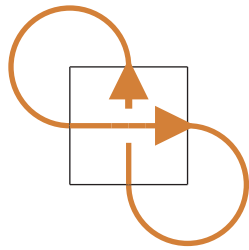


**2** T-tiles contain two quarter circles.



**3** Replacing one of the X-tiles in Figure 1 with a T-tile. The orientation of the T-tile determines whether the band is cut into two pieces. (a) The result is a single continuous band. (b) Two separate bands.

**4** The simplest knot. (a) The knot. (b) The directed skeleton.



**5** Replacing the intersection. (a) If you join arrowheads to tails, you get two loops. (b) If you join arrowheads together, you get a single band.

If you reach your starting point and you haven't marked all the pieces, you increment the band count and start again with an unmarked piece of band. Keep doing this until the entire pattern is marked, and you'll have your band count.

When I first started hand-drawing knots, this is exactly what I did, using a set of colored pencils. I'd trace over the design again and again, starting over again after each change to see if the number of bands had changed (or not changed) the way I'd desired.

If you have a computer helping you (such as my Knot Assistant from my last column), then the computer can solve this problem for you (if it's fast enough). If a change has an undesired effect, you undo it and put down the T-tile the other way. But I'd still like to point to a tile and declare my intention, say left-clicking to preserve the band count or right-clicking to cause it to increase or decrease.

I stumbled across the way to do this while working with a band design made up of arrows. It's really pretty simple. Let's look at the very simplest piece of overlapping knotwork: a single intersection, shown in Figure 4a. Suppose the band is a pipe carrying water. You can pick an arbitrary direction for one piece of the band, then just follow the flow of water forward, marking the segments as you go. Figure 4b shows the result, which I call the *directed skeleton*.

Figure 5 shows the result of replacing this one intersection with each of the possible ways to reroute the path. One choice cuts the band into two loops, the other preserves it as only one. You can see that if you join up the arrowheads, you get a single band. If you join an arrowhead to a tail, you get two bands.

Each intersection of any piece of knotwork can be considered nothing more than Figure 4. All the rest of the design beyond the intersection boils down to the loops outside the intersection. As long as the labeling stays consistent throughout the
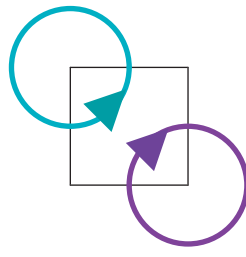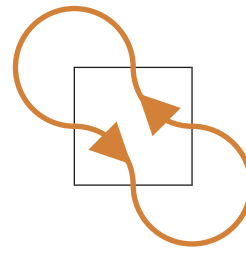
design, you can simply use the arrowhead rule to replace any intersection.

One "gotcha" to keep in mind is that after a replacement, you'll always need to relabel at least some of the band because now the water is flowing in the opposite direction. A computer program can do this easily and automatically. Even when working with pencil and paper, it's easier to do this relabeling once after each replacement than to trace the whole pattern after each change to see if you've accidentally cut the band in two.

## Snakes and tiles

If you replace each of the intersections one by one until they're all gone, always maintaining a single continuous band, you end up with what I call a *snake*.

Figure 6 shows the process of creating a snake for a simple knot. Pick any X-tile at random and replace it with a T-tile. You then have to catch any other replacements that are forced as a result in order to maintain the
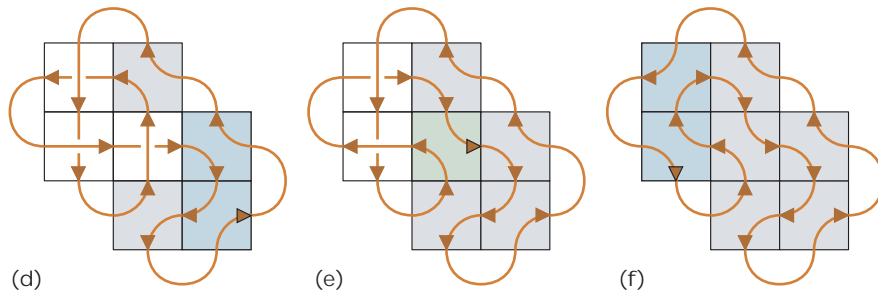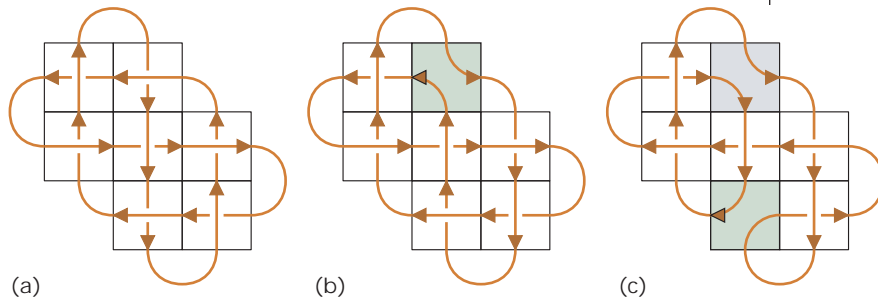


**6** Building a snake from a knot. At each step, the green tile is the one that has been replaced. Blue tiles result from the forced changes from the previous step. Gray tiles are those that have been modified so far.

**7** The six snakes that can be built from the knot of Figure 6a.
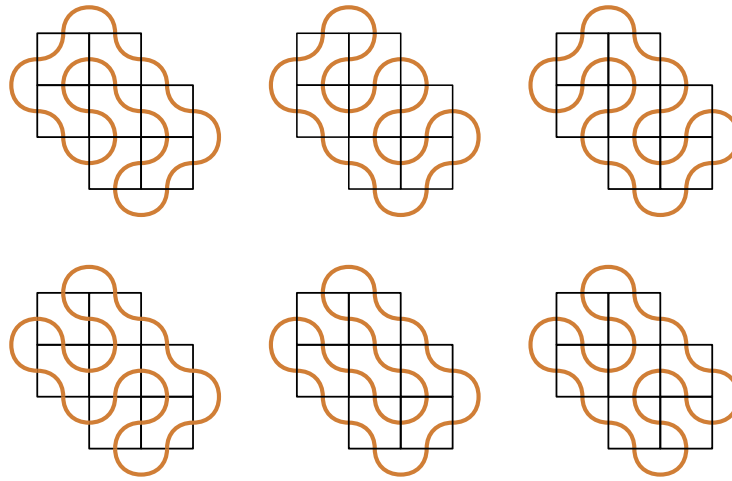
single band. If you have any X-tiles left over, you can pick another one at random.

The order in which you select the intersections to replace determines the snake's shape. Figure 7 shows all the snakes for this simple seven-cell knot. More complex knots will have many more possibilities.

How can we turn the process around, adding an intersection to a snake without increasing the band count? It's simple. Find any T-tile where the bands are going in opposite directions and replace it with an X-tile.

As I've discussed in earlier columns, one way to create ordered layouts of tiles is to mark them so that they can only be connected in legal ways. Celtic knotwork and snakes are no exception. Figuring out the right tiles for Celtic knotwork was harder than I thought it would be, because the only way to keep the directional flow correct is to rely on the outer-border tiles to force proper patterns. Some tiling methods, like those for many asymmetrical patterns, allow you to choose from more than one tile for some steps. This can lead you down dead ends. You add tiles according to the rules, but then you reach a state where you just can't add any more.

Then you have to pull up some of the tiles and backtrack, making a new choice and working forward again. My tiles are like that too. The good news is that if you complete a pattern, you have a complete and properly woven skeleton.

Figure 8 shows my Celtic knotwork tiles. Assemble the pieces like a jigsaw puzzle. Start with one of the X-tiles from Figure 8a, and add corner pieces from Figure 8b. When you want to start changing things, replace any intersection with one of the T-tiles in Figure 8c, depending on whether you want to keep the band count or increment it, and then replace the other tiles as needed to get a consistent knot. During this retiling step, you may need to replace some of the X-tiles with the other form of X-tile, and some of the T-tiles in Figure 8c with one of the T-tiles in Figure 8d. These tiles are printed on only one side—you can rotate them, but don't flip them over.

Figure 9a shows a skeleton built with these tiles and then a couple of replacements of X-tiles with T-tiles. Figure 10 shows the knotwork version of this skeleton.

## New grids, new bands

In my last column I discussed Iain Bain's three-grid approach for constructing knotwork. That approach always leads to knots built on rectangular grids, which is a pretty broad class of knots, but doesn't come close to covering all the possibilities.

The first generalization is to keep the four-sided cells, but lose the checkerboard nature of the square grid. Figure 11 shows how to create a triangular-shaped net. Figure 12 shows a circular band. The cells still have four sides each, but the sides are curved and the connectivity differs from that for a square grid.

Using circles breaks free of the square grid. To com-
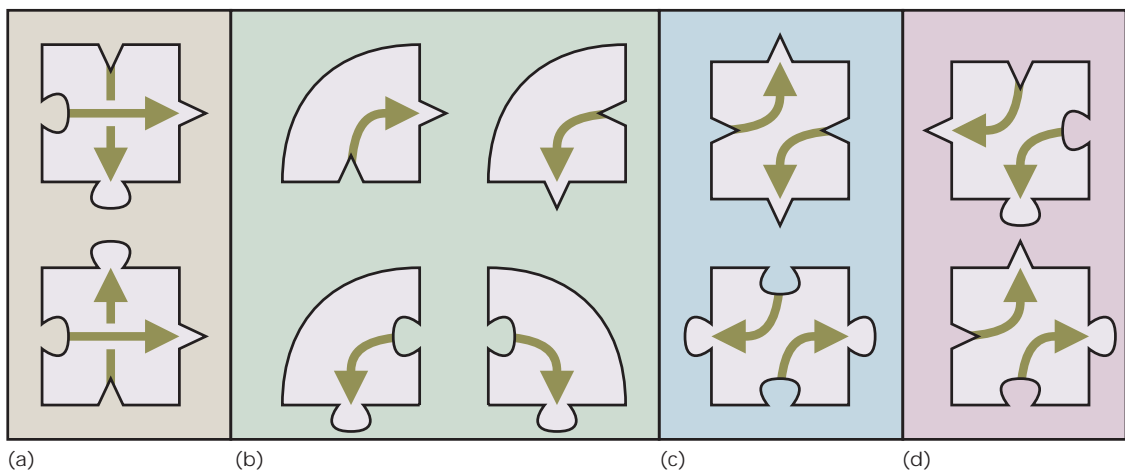
**8** The Celtic Knot Jigsaw pieces printed on only one side. You can rotate them, but don't flip them over. (a) The X-tiles. Use one of these to build the initial knotwork. (b) Corners. (c) Initial T-tiles. Replace an X-tile with one of these. (d) Other T-tiles. You may need to replace T-tiles with these when retiling.

plement last column's Knot Assistant, I wrote a Circular Knot Assistant. Figure 13 shows a screen shot of the Circular Knot Assistant after I've created three circles. I've thickened the lines a little more than usual so that they'll read clearly in print.

Actually, these aren't circles at all, but sets of Bezier curves. Each "circle" consists of four 90-degree Bezier arcs. I fooled around with Bezier curves for a while and found a good arc can be made with the following recipe. If the center of the circle lies at the origin and the circle has radius 1, then the four Bezier points for the northeast arc are $(1, 0)$, $(1, s)$, $(s, 1)$, $(0, 1)$, where $s = 0.543474$. The maximum distance from this curve to the circle is less than 0.005 units. The other three arcs are just reflections of this one.

The general idea is that you draw circles. Then you select, modify, and join up segments. When you select a segment, the Circular Knot Assistant marks it in black and draws a box at the nearest point on
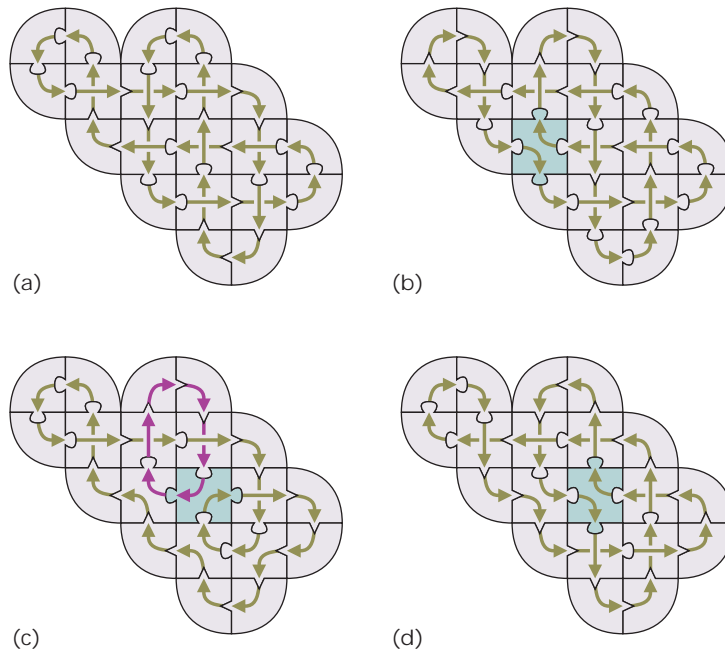


**9** (a) A knot built with the tiles of Figure 8. (b) Replacing an X-tile with a T-tile. (c) A second replacement, creating two bands. (d) A second replacement conserving the single band count.
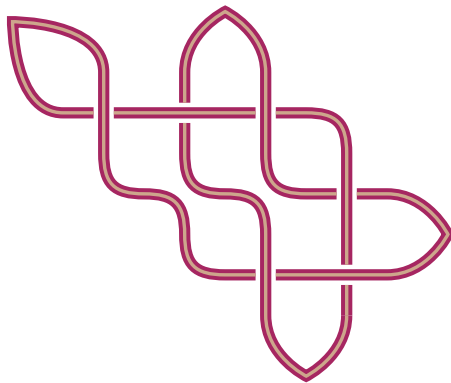


**10** The knotwork of the tiled pattern in Figure 9d.
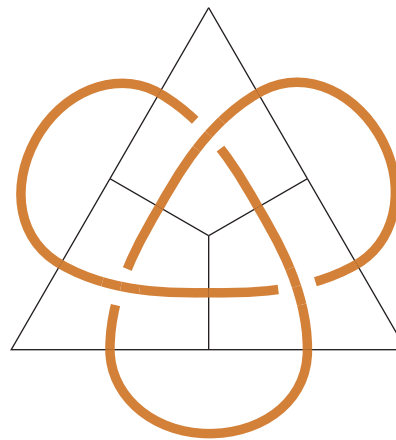


**11** A triangular net. Note that the cells still have four sides each.
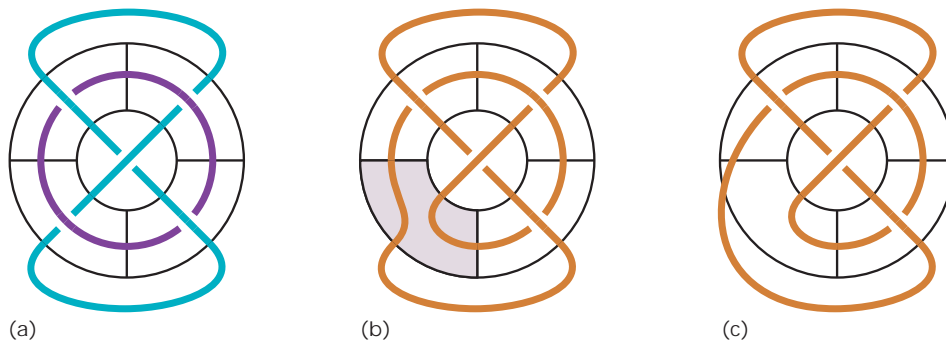


**12** A circular net. Each cell has four sides, including the central cell (each side is a 90-degree arc). (a) All X-tiles create two bands. (b) Replacing an X-tile with a T-tile, reducing the band count to 1. (c) Smoothing out the result.

**13** A screen shot of the Circular Knot Asssistant.
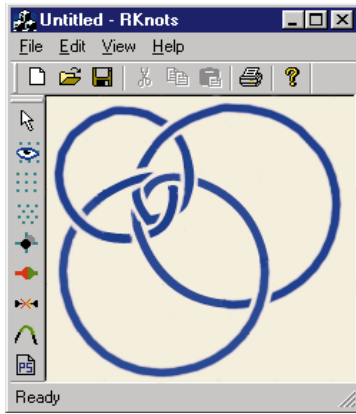


the curve to where you clicked, which I call the *selection point*. This ensures you've selected the correct end of the curve when you want to join it up with another one. If you hold down the shift key, you can select multiple curves.

The top row of buttons in Figures 13 and 14a are standard Windows buttons. Along the left are my own buttons. At the top is a selection tool. Below that is an icon for viewing an underlying grid, with buttons to select a rectangular or hexagonal grid. The next button down toggles snapping to the grid. The red/green button cuts a segment at the selection point. The next button deletes the selected segment. The hairpin is my Blendo tool. It fits a new Bezier curve between two selected curves, using the endpoints nearest their respective selections. The button at the bottom triggers the output of the knot in Postscript.

Figure 14 shows a few steps in constructing a simple knot.

I've been taking the output of my Circular Knot Assistant and using it as a basis for drawings by hand, which I then bring into Photoshop for enhancement. Figure 15 shows three of my knots done this way.

These nonsquare grids open up the door to creating more complex arrangements of knotwork. In my last column, I promised we'd move on to three-dimensional knots. We'll do that next time, leaping into 3D in my first column of the 3rd millennium. ∎
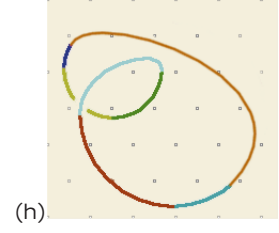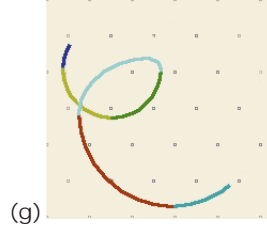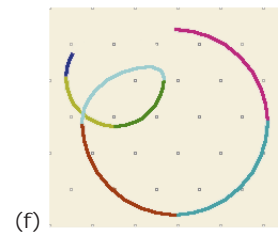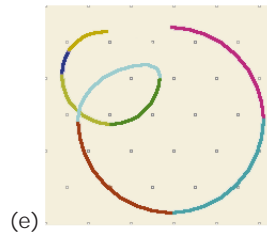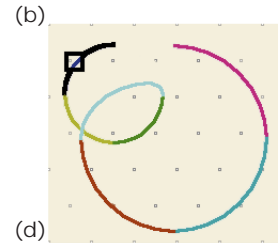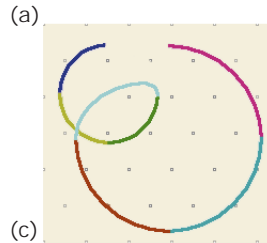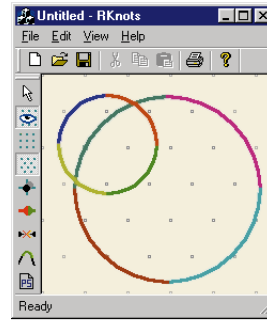
**14** Developing a simple circular knot. (a) Draw two circles. (b) Delete two arcs. (c) Use the blend tool to insert a new arc. (d) Select an arc. (e) Cut in two. (f) Delete one of the new segments. (g) Trim another segment. (h) Join the arcs, and turn on automatic overlapping.
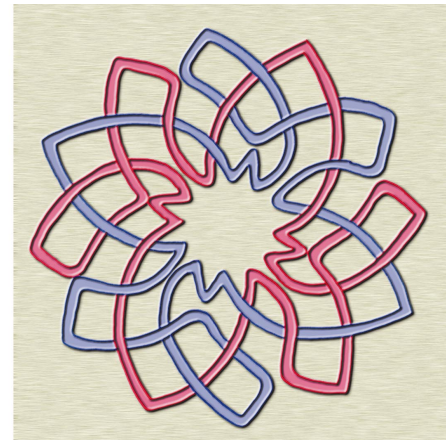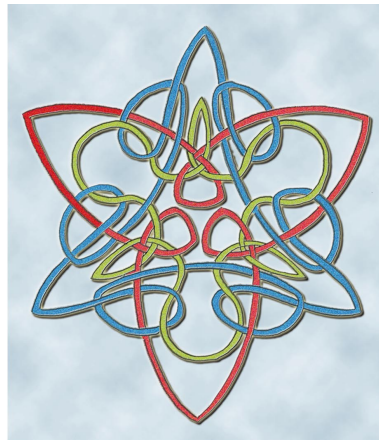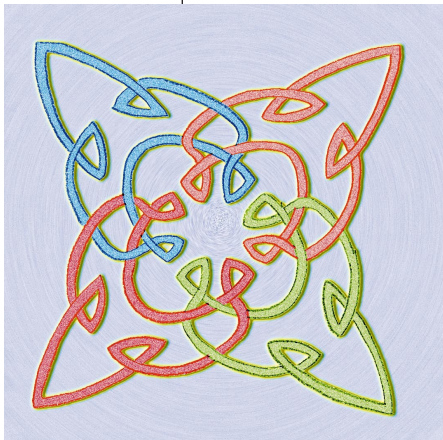


**15** Hand-drawn knots based on output from the Circular Knot Assistant.

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