Two Derivations of the Angular Interpolation Formula

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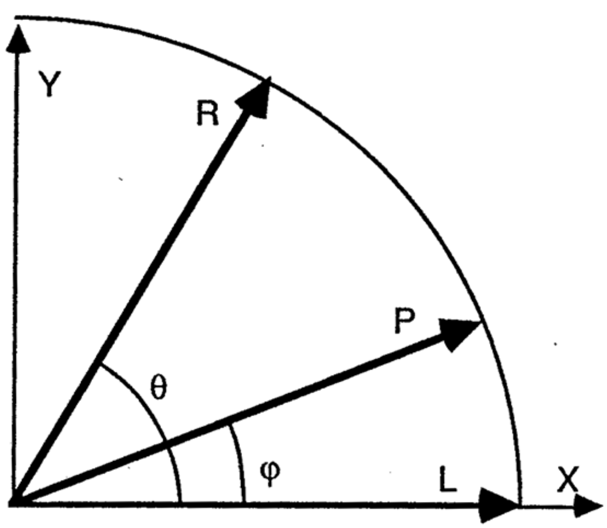
Abstract

Given two unit vectors **L** and **R**, and a scalar α between 0 and 1, we wish to find the vector **P** between **L** and **R** such that $\angle(\mathbf{L},\mathbf{P}) = \alpha\angle(\mathbf{L},\mathbf{R})$ (so as α ranges from 0 to 1, **P** interpolates from **L** to **R** around the unit circle). The solution is the angular interpolation formula $\mathbf{P} = \beta_1\mathbf{L} + \beta_2\mathbf{R}$, where $\beta_1=\sin((1-\alpha)\theta)/\sin\theta$, $\beta_2=\sin(\alpha\theta)/\sin\theta$, and $\theta=\cos^{-1}(\mathbf{L}\bullet\mathbf{R})$. Angular interpolation has applications including quaternion rotation [Shoemake85] (where the formula is presented without derivation), and circular shading [Glassner87]. The two constructions shown here were developed by the authors simultaneously but independently. We present both derivations (rather than just one), in the spirit of [Heckbert87]: to show two contrasting solutions to a geometric problem.

Notation

Since **L**, **R**, and **P** are all coplanar, we may write **P** as a linear combination of **L** and **R** (assuming **L** and **R** are not colinear): $\mathbf{P} = \beta_1\mathbf{L} + \beta_2\mathbf{R}$. If **L** and **R** *are* colinear then it is undetermined through which semicircle **P** should pass, and the interpolation should be broken into two smaller pieces. We label the angle between **L** and **R** with θ, and the angle between **L** and **P** with φ. Thus $\varphi = \alpha\theta$.

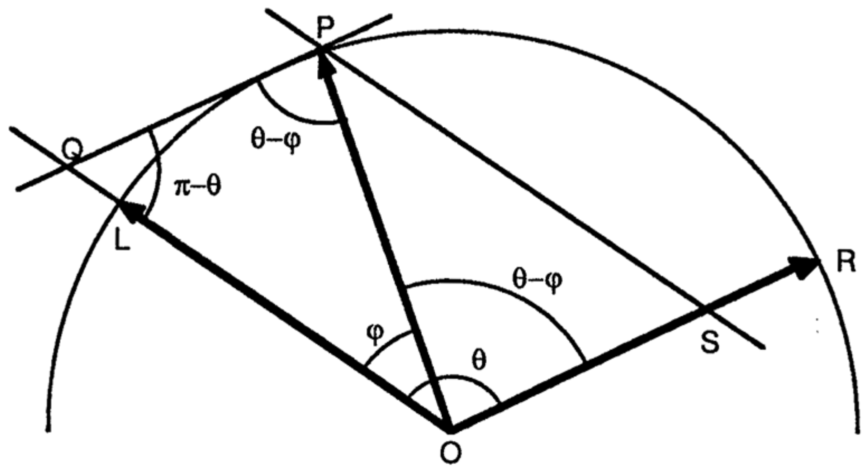
Derivation 1



In this diagram we have rigidly rotated the vectors **L** and **R** so that **L** lies on the Cartesian X axis. From the diagram, we see that $\mathbf{P} = (\cos\varphi, \sin\varphi)$, $\mathbf{R} = (\cos\theta, \sin\theta)$, and $\mathbf{L} = (1, 0)$. Solving the linear relation first for the y component of **P**, we find $\mathbf{P}_y = \beta_1\mathbf{L}_y + \beta_2\mathbf{R}_y = \beta_2\mathbf{R}_y$. We can rewrite this to obtain $\beta_2 = \mathbf{P}_y/\mathbf{R}_y$. Solving for the x component, $\mathbf{P}_x = \beta_1\mathbf{L}_x + \beta_2\mathbf{R}_x = \beta_1 + \beta_2\mathbf{R}_x$. We rewrite this as $\beta_1 = \mathbf{P}_x - \beta_2\mathbf{R}_x$. Substituting the actual co-ordinates of the points gives us

$$\beta_2 = \frac{\sin\varphi}{\sin\theta} \qquad \beta_1 = \cos\varphi - \beta_2\cos\theta$$

Derivation 2



In the diagram above, **P** is shown as the sum of β_1**L** and β_2**R**. Thus OQPS is a parallelogram, in which $|OQ| = \beta_1$, $|OS| = \beta_2$, $|OP| \equiv 1$, and $|QP| = |OS|$. Note that QP is parallel to OS, and in general will not be tangent to the unit circle. In ΔOPQ, $\angle O=\varphi$, $\angle P=\theta-\varphi$, and $\angle Q=\pi-\theta$. We can find β_1 and β_2 with the law of sines in ΔOPQ (recall that $\sin(\pi-\theta) = \sin(\theta)$): $\frac{\sin(\pi-\theta)}{1} = \frac{\sin\theta}{1} = \frac{\sin\varphi}{\beta_2} = \frac{\sin(\theta-\varphi)}{\beta_1}$. Solving for β_1 and β_2 we have:

$$\beta_1 = \frac{\sin(\theta-\varphi)}{\sin\theta} \qquad \beta_2 = \frac{\sin\varphi}{\sin\theta}$$

Equivalence and Computation

Of course the two different expressions for β_1 from derivations 1 and 2 must be equivalent. To demonstrate this, recall the trig identity $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$. Then we can rewrite the first expression as

$$\beta_1 = \cos\varphi - \beta_2\cos\theta = \cos\varphi - \frac{\sin(\theta-\varphi)}{\sin\theta}\cos\theta = \frac{\sin\theta\cos\varphi - \sin\varphi\cos\theta}{\sin\theta} = \frac{\sin(\theta-\varphi)}{\sin\theta}$$

We may efficiently compute β_1 and β_2 using the forms in Derivation 2. Given **L**, **R**, and α:

1. $\theta = \cos^{-1}(\mathbf{L}\bullet\mathbf{R})$
2. $\varphi=\alpha\theta$
3. $f = 1/\sin(\theta)$
4. $\beta_1 = f \times \sin(\theta-\varphi)$
5. $\beta_2 = f \times \sin(\varphi)$

References

[Shoemake85] K. Shoemake, "Animating Rotations with Quaternion Curves", *Computer Graphics* (19)3 (Proceedings of Siggraph '85), July 1985
[Glassner87] A. Glassner, "Rendering Rounder Polygons", in preparation
[Heckbert87] P. Heckbert, "Derivation of Refraction Formulas", *Introduction to Ray Tracing* course notes, Siggraph '87, July 1987